MAT 2125: FIELD AND ORDER NOTES

1. Overview

I think that a few people are a bit concerned about some of the following:

(1) Do I really need to use these axioms in every line of every calculation for the rest of the course?
(2) There are lots of "obvious" facts that I can't find in the course notes. What should I do?
(3) The course notes don't prove that $1 > 0$. Where do I find this particular proof?
(4) Some of these arguments are weird. How was I supposed to come up with these tricks?
(5) What am I supposed to be able to do with the axiom stuff?

To give quick responses in order,

(1) No. After the first short while, we're just going to use obvious facts about the algebraic and order structure of $\mathbb{Q}$ without discussion. We'll also use the fact that $\mathbb{Q}$ is sitting inside of $\mathbb{R}$
(2) There will not be any point in the course where you need to cite "obvious" facts about fields or orders that are missing from the course notes. To provide some context: it turns out to be a huge amount of work to build up the details of $\mathbb{R}$ from nothing, and we're not going to do all that work in this course. We'll still pay a lot of attention to things that aren't in $\mathbb{Q}$, such as $\sqrt{2}$.
(3) This proof, and many other "obvious" facts about the order structure of fields, are contained in the next section of this document. **Note:** this is still far from a comprehensive list. I don't prove many "obvious" facts about fields (e.g. that $(mn)^{-1} = m^{-1}n^{-1}$). As I say above, we're mostly going to skip this.
(4) The short answer is, you're not. There are several places, such as Equality (2.2), where you have to write down some identity that is not really suggested by the question. This is genuinely hard, and we're not going to focus on that sort of thing in this course. See the last section for a very quick discussion of what we are going to focus on.
(5) The short version is: you're supposed to be able to do the homework questions, and a small number of minor variants. You're also supposed

to be aware that this axiomatic stuff is underpinning our later calculations, even if we don't exhaustively document every manipulation that we perform.

The rest of the document mostly contains proofs. I want to emphasize that you are *not* expected to read these carefully and/or memorize them - I'm sending them only because some people have requested them.

You could use them as practice problems with solutions, if you're looking for practice problems. I broke up long arguments into many pieces so that all of them are somewhat accessible, though obviously some are longer/trickier than others. I'll note that I didn't always go for the most efficient proof - in a few places I took a detour to prove something that seemed interesting, and in one place I took a detour to avoid a homework problem.

2. Proving "Obvious" Inequalities for $\mathbb{Q}$

We develop a collection of "obvious" inequalities for an ordered field from just the order axioms. I try to use "lemma" to indicate intermediate results that are unlikely to be used directly.

In this section, I denote by F an ordered field that is assumed to have at least one nonzero element. Throughout, I use the notational shorthands from the video. For example,"2" is shorthand for "$1 + 1$" and $\frac{2}{5}$ is shorthand for 2×5^{-1}.

Theorem 2.1. *For all $x \in F$, $0x = 0$.*

Proof. We have

$$\begin{aligned} 0x &\overset{A3}{=} (0+0)x \\ &\overset{AM1}{=} 0x + 0x. \end{aligned}$$

Thus, by A4, $0 = 0x$. $\square$

Theorem 2.2. *For all $x \in F$, $(-a) = (-1)(a)$.*

Proof. We have

$$\begin{aligned} 0 &\overset{\text{Thm. } 2.1}{=} 0x \\ &\overset{A4}{=} (-1+1)x \\ &\overset{AM}{=} (-1)x + (1)x \\ &\overset{M3}{=} (-1)x + x. \end{aligned}$$

$\square$

Theorem 2.3. *We have* $(-1)^2 = 1$.

Proof. Write

$$\begin{aligned}
0 &\overset{\text{Thm. 2.1}}{=} 0^2 \\
&\overset{A4}{=} (1-1)^2 \\
&\overset{AM1}{=} (1)(1) + (1)(-1) + (-1)(1) + (-1)^2 \\
&\overset{M3}{=} 1 - 1 - 1 + (-1)^2 \\
&\overset{A4}{=} -1 + (-1)^2.
\end{aligned}$$

Rearranging gives the desired equality. □

Theorem 1. *For all* $x \in \mathbb{R}$, $x^2 \geq 0$.

Proof. We consider two cases: $x \geq 0$ and $x < 0$.

(1) If $x \geq 0$, then

$$\begin{aligned}
x^2 &= (x)(x) \\
&\overset{O4}{\geq} (0)(x) \\
&\overset{\text{Thm. 2.1}}{=} 0.
\end{aligned} \tag{2.1}$$

(2) If $x < 0$, then

$$\begin{aligned}
x^2 &\overset{M3}{=} (1)x^2 \\
&\overset{\text{Thm. 2.3}}{=} (-1)^2 x^2 \\
&\overset{M1}{=} ((-1)(x))((-1)(x)) \\
&\overset{\text{Thm. 2.2}}{=} (-x)^2 \\
&\overset{\text{Ineq. (2.1)}}{\geq} 0.
\end{aligned}$$

Thus we have the desired inequality in both cases. □

Lemma 2.4. *We have* $1 > 0$.

Proof. By O1, it is enough to rule out the possibility that $1 = 0$ and the possibility that $1 < 0$.

- **Ruling out** $1 = 0$**:** Assume otherwise, so that $1 = 0$. Let $x \in F$ be a nonzero element of F. Then

$$x \overset{M3}{=} 1x$$

$$
\begin{aligned}
&\overset{\text{Assumption}}{=} 0x \\
&\overset{\text{Thm. 2.1}}{=} 0.
\end{aligned}
$$

But this contradicts the fact that $x \neq 0$. Having derived a contradiction, we conclude that the assumption $1 = 0$ must be false.

- **Ruling out $1 < 0$:** We have

$$
\begin{aligned}
1 &\overset{M3}{=} 1^2 \\
&\overset{\text{Thm. 1}}{\geq} 0.
\end{aligned}
$$

Having ruled out both of these possibilities, we conclude that $1 > 0$. □

Lemma 2.5. *For all $n \in \mathbb{N}$, $n > 0$.*

Proof. We prove this by induction. The base case is Lemma 2.4. To prove the inductive step, fix $n \in \mathbb{N}$ and assume $n > 0$; we must show that $n + 1 > 0$.

Since $n > 0$, we have by O3 that $n + 1 > 0 + 1$. We continue:

$$
\begin{aligned}
n + 1 &> 0 + 1 \\
&\overset{A3}{=} 1 \\
&\overset{\text{Lemma 2.4}}{>} 0.
\end{aligned}
$$

This completes the proof. □

Lemma 2.6. *Consider $a, b \in F$. We have $ab = 0$ if and only if at least one of a, b are equal to 0.*

Proof. One direction is easy: if either a, b are equal to 0, then $ab = 0$ by Theorem 2.1.

We prove the other direction by contradiction. Assume there exist $a, b \neq 0$ with $ab = 0$. We then calculate

$$
\begin{aligned}
0 &\overset{\text{Thm. 2.1}}{=} 0b^{-1} \\
&\overset{\text{Assumption}}{=} (ab)b^{-1} \\
&\overset{M4}{=} a1 \\
&\overset{M3}{=} a.
\end{aligned}
$$

But this contradicts the assumption that $a \neq 0$. □

Theorem 2.7. *For all $m, n \in \mathbb{N}$, $\frac{m}{n} > 0$.*

Proof. We can write
$$\frac{m}{n} \overset{M3}{=} (m)(n)(n^{-1})^2. \tag{2.2}$$
We note that $(n^{-1})^2 > 0$ by Lemma 1 (which says it is nonnegative), Lemma 2.6 (which rules out the possibility that it is 0), and O1. We note that $m, n > 0$ by Lemma 2.5. Thus, all three elements of the product in Equation (2.2) are strictly greater than 0. By 04, the product must also be strictly greater than 0. □

Lemma 2.8. *If $a > 0$, then $-a < 0$. Similarly, if $a < 0$, then $-a > 0$.*

Proof. We have $-a = (-1)(a)$ by Lemma 2.2, and so $-a \neq 0$ by Lemma 2.6.

We now proceed by contradiction. Assume $-a \geq 0$. Since we just showed $-a \neq 0$, this would imply $-a > 0$. But then by O3,
$$a + -a > 0 + 0 \overset{A3}{=} 0.$$
But applying A4 to the left-hand side of this inequality, we have
$$0 = a + -a > 0,$$
which is a contradiction.

The proof of the second part is essentially identical. □

Theorem 2.9. *Fix $a, b, c \in F$ with $a < b$ and $c < 0$. Then $ac > bc$.*

Proof. We prove this by contradiction: assume $ac \leq bc$ for some $a < b$ and $c < 0$. We first check that $ac = bc$ is impossible. We have $c \neq 0$, and so $ac = bc$ would imply
$$\begin{aligned} a &\overset{M4}{=} acc^{-1} \\ &\overset{\text{Assumption}}{=} bcc^{-1} \\ &\overset{M4}{=} b, \end{aligned}$$
contradicting the assumption that $a < b$.

So we now have that $ac < bc$. By Lemma 2.8, $-c > 0$. Thus, by O4,
$$a(-c) < b(-c).$$
Adding this to $ac < bc$ and applying O3, we have
$$ac + a(-c) < bc + b(-c).$$
Applying AM1 and then A4, we have $ac + a(-c) = a(c - c) = 0$, and similarly $bc + b(-c) = 0$. Thus, we have
$$0 < 0,$$

which is clearly a contradiction.

$\square$

3. Tricks and This Course

I think that lots of the proofs in the last section "feel bad" because they are quite ad-hoc. Where did Equation (2.2) come from? Who would ever think to rewrite 0 as $(1-1)^2$, like in the proof of Theorem 2.3?

I feel the same way. These sorts of out-of-nowhere arguments will be much less common in the rest of the course. Instead, there will be a fairly small collection of techniques that get used over and over again. Since I'm writing this near the start of the course, we haven't seen many of these yet, and we haven't seen any repeated. I'll write a few pointers for now, but we'll discuss this much more in class once we've accumulated a few examples for each technique.

The first "major" technique is the telescoping sum identity: if you have a sequence $x_1, x_2, \ldots, x_n$, then

$$x_n - x_1 = \sum_{i=1}^{n-1} (x_{i+1} - x_i). \tag{3.1}$$

By the triangle inequality, this gives:

$$|x_n - x_1| \leq \sum_{i=1}^{n-1} |x_{i+1} - x_i|.$$

At first glance this seems a bit dopey: if you only see x_n and x_1, where do $x_2, \ldots, x_{n-1}$ come from?

The short answer is that, at least in this class, they're normally things that are basically in the question. The first time this telescoping sum idea appears is in the proof of Proposition 2.5 of the lecture notes.

The closest thing we've seen to a recurring technique is the idea that strict inequalities always come with a bit of wiggle room. For example, see the following proof:

Lemma 3.1. *Let* $S = \{x \in \mathbb{R} : x < 1\}$. *Then* $\sup(S) = 1$.

Proof. First, we note that 1 is an upper bound on S: if $x \in S$, then $x < 1$ by definition.

Next, we need to show that 1 is the *least* upper bound. We do this by contradiction. Assume there exists some $x < 1$ that is also an upper bound.

Then, as shown in Lemma 1.2 of the course notes,

$$x < \frac{x+1}{2} < 1.$$

But this means that $\frac{x+1}{2} \in S$, so x is not an upper bound on S.

We've shown that 1 is an upper bound on S, and we've shown that there doesn't exist a strictly smaller upper bound. This completes the proof. $\square$

At some point in the argument, we got to assume that $x < 1$. We then notice that there's some wiggle room: since $x < 1$, there must be a bunch of things in between x and 1. The same idea gets used in many places, and is especially powerful in combination with Question 3.3 of HW1.