MAT 2125 – Exercises

1. Let $a, b \in \mathbb{R}$ and suppose that $a \leq b + \varepsilon$ for all $\varepsilon > 0$. Show that $a \leq b$.

2. Let $c > 0$ be a real number.

   (a) If $c > 1$, show that $c^n \geq c$ for all $n \in \mathbb{N}$ and that $c^n > 1$ if $n > 1$.

   (b) If $0 < c < 1$, show that $c^n \leq c$ for all $n \in \mathbb{N}$ and that $c^n < 1$ if $n > 1$.

3. Let $c > 0$ be a real number.

   (a) If $c > 1$ and $m, n \in \mathbb{N}$, show that $c^m > c^n$ if and only if $m > n$.

   (b) If $0 < c < 1$ and $m, n \in \mathbb{N}$, show that $c^m > c^n$ if and only if $m < n$.

4. Let $S_2 = \{x \in \mathbb{R} \mid x > 0\}$. Does S_2 have lower bounds? Does S_2 have upper bounds? Does $\inf S_2$ exist? Does $\sup S_2$ exist? Prove your statements.

5. Let $S_4 = \left\{1 - \frac{(-1)^n}{n} \mid n \in \mathbb{N}\right\}$. Find $\inf S_4$ and $\sup S_4$.

6. Let $S \subseteq \mathbb{R}$ be non-empty. Show that if $u = \sup S$ exists, then for every number $n \in \mathbb{N}$ the number $u - \frac{1}{n}$ is not an upper bound of S, but the number $u + \frac{1}{n}$ is.

7. If $S = \left\{\frac{1}{n} - \frac{1}{m} \mid m, n \in \mathbb{N}\right\}$, find $\inf S$ and $\sup S$.

8. Let X be a non-empty set and let $f : X \to \mathbb{R}$ have bounded range in $\mathbb{R}$. If $a \in \mathbb{R}$, show that

$$\sup\{a + f(x) : x \in X\} = a + \sup\{f(x) : x \in X\}$$
$$\inf\{a + f(x) : x \in X\} = a + \inf\{f(x) : x \in X\}.$$

9. Let A and B be bounded non-empty subsets of $\mathbb{R}$, and let

$$A + B = \{a + b \mid a \in A, b \in B\}.$$

   Prove that $\sup(A + B) = \sup A + \sup B$ and $\inf(A + B) = \inf A + \inf B$.

10. Let X be a non-empty set and let $f, g : X \to \mathbb{R}$ have bounded range in $\mathbb{R}$. Show that

$$\sup\{f(x) + g(x) \mid x \in X\} \leq \sup\{f(x) \mid x \in X\} + \sup\{g(x) \mid x \in X\}$$
$$\inf\{f(x) \mid x \in X\} + \inf\{g(x) \mid x \in X\} \leq \inf\{f(x) + g(x) \mid x \in X\}.$$

11. Let X and Y be non-empty sets and let $h : X \times Y \to \mathbb{R}$ have bounded range in $\mathbb{R}$. Let $F : X \to \mathbb{R}$ and $G : Y \to \mathbb{R}$ be defined by

$$F(x) = \sup\{h(x, y) \mid y \in Y\} \quad \text{and} \quad G(y) = \sup\{h(x, y) \mid x \in X\}.$$

Show that

$$\begin{aligned} \sup\{h(x, y) \mid (x, y) \in X \times Y\} &= \sup\{F(x) \mid x \in X\} \\ &= \sup\{G(y) \mid y \in Y\}. \end{aligned}$$

12. Show there exists a positive real number u such that $u^2 = 3$.

13. Show there exists a positive real number u such that $u^3 = 2$.

14. Let $S \subseteq \mathbb{R}$ and suppose that $s^* = \sup S$ belongs to S. If $u \notin S$, show that $\sup(S \cup \{u\}) = \sup\{s^*, u\}$.

15. Show that a non-empty finite set $S \subseteq \mathbb{R}$ contains its supremum.

16. If $S \subseteq \mathbb{R}$ is a non-empty bounded set and $I_S = [\inf S, \sup S]$, show that $S \subseteq I_S$. Moreover, if J is any closed bounded interval of $\mathbb{R}$ such that $S \subseteq J$, show that $I_S \subseteq J$.

17. Prove that if $K_n = (n, \infty)$ for $n \in \mathbb{N}$, then

$$\bigcap_{n \in \mathbb{N}} K_n = \varnothing.$$

18. If S is finite and $s^* \notin S$, show $S \cup \{s^*\}$ is finite.

19. The first few terms of a sequence (x_n) are given below. Assuming that the "natural pattern" indicated by these terms persists, give a formula for the nth term x_n.

    (a) $(5, 7, 9, 11, \ldots)$;

    (b) $(\frac{1}{2}, -\frac{1}{4}, \frac{1}{8}, -\frac{1}{16}, \ldots)$;

    (c) $(\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots)$;

    (d) $(1, 4, 9, 16, \ldots)$.

20. Use the definition of the limit of a sequence to establish the following limits.

    (a) $\displaystyle\lim_{n\to\infty} \left(\frac{1}{n^2 + 1}\right) = 0$;

    (b) $\displaystyle\lim_{n\to\infty} \left(\frac{2n}{n + 1}\right) = 2$;

    (c) $\displaystyle\lim_{n\to\infty} \left(\frac{3n + 1}{2n + 5}\right) = \frac{3}{2}$, and

    (d) $\displaystyle\lim_{n\to\infty} \left(\frac{n^2 - 1}{2n^2 + 3}\right) = \frac{1}{2}$.

21. Show that

(a) $\lim_{n\to\infty}\left(\frac{1}{\sqrt{n+7}}\right)=0$;

(b) $\lim_{n\to\infty}\left(\frac{2n}{n+2}\right)=2$;

(c) $\lim_{n\to\infty}\left(\frac{\sqrt{n}}{n+1}\right)=0$, and

(d) $\lim_{n\to\infty}\left(\frac{(-1)^n n}{n^2+1}\right)=0$.

22. Show that $\lim_{n\to\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)=0$.

23. Find the limit of the following sequences:

(a) $\lim_{n\to\infty}\left(\left(2+\frac{1}{n}\right)^2\right)$;

(b) $\lim_{n\to\infty}\left(\frac{(-1)^n}{n+2}\right)$;

(c) $\lim_{n\to\infty}\left(\frac{\sqrt{n}-1}{\sqrt{n}+1}\right)$, and

(d) $\lim_{n\to\infty}\left(\frac{n+1}{n\sqrt{n}}\right)$.

24. Let $y_n=\sqrt{n+1}-\sqrt{n}$. Show that (y_n) and $(\sqrt{n}y_n)$ converge.

25. Find the limit of the following sequences:

noitemsep $\lim_{n\to\infty}\frac{\sin(n^2+212)}{n}$;

noiitemsep $\lim_{n\to\infty} q^n$, if $|q|<1$;

noiiitemsep $\lim_{n\to\infty}\sqrt[n]{n}$;

noivtemsep $\lim_{n\to\infty}\frac{n!}{n^n}$, and

novtemsep $\lim_{n\to\infty}\sqrt[n]{3^n+5^n}$.

26. Let (x_n) be a sequence of positive real numbers such that

$$\lim_{n\to\infty} x_n^{1/n}=L<1.$$

Show $\exists r\in(0,1)$ such that $0<x_n<r^n$ for all sufficiently large $n\in\mathbb{N}$.

Use this result to show that

$$\lim_{n\to\infty} x_n=0.$$

27. Give an example of a convergent (resp. divergent) sequence (x_n) of positive real numbers with $x_n^{1/n} \to 1$.

28. Let $x_1 = 1$, $x_{n+1} = \sqrt{2 + x_n}$ for $n \in \mathbb{N}$. Show that (x_n) converges and find the limit.

29. Let $x_n = \displaystyle\sum_{k=1}^{n} \frac{1}{k^2}$ for all $n \in \mathbb{N}$.

    Show that (x_n) is increasing and bounded above.

30. Show that $c^{1/n} \to 1$ if $0 < c < 1$.

31. Let (x_n) be a bounded sequence.

    For each $n \in \mathbb{N}$, let $s_n = \sup\{x_k : k \geq n\}$. If $S = \inf\{s_n\}$, show that there is a subsequence of (x_n) that converges to S.

32. Suppose that $x_n \geq 0$ for all $n \in \mathbb{N}$ and that $((-1)^n x_n)$ converges. Show that (x_n) converges.

33. Show that if (x_n) is unbounded, then there exists a subsequence (x_{n_k}) such that $1/x_{n_k} \to 0$.

34. If $x_n = \frac{(-1)^n}{n}$, find the convergent subsequence in the proof of the Bolzano-Weierstrass theorem, with $I_1 = [-1, 1]$.

35. Show directly that a bounded increasing sequence is Cauchy.

36. If $0 < r < 1$ and $|x_{n+1} - x_n| < r^n$ for all $n \in \mathbb{N}$, show that (x_n) is Cauchy.

37. If $x_1 < x_2$ and $x_n = \frac{1}{2}(x_{n-1} + x_{n-2})$ for all $n \in \mathbb{N}$, show that (x_n) is convergent and compute its limit.

38. Let $f : \mathbb{R} \to \mathbb{R}$ and let $c \in \mathbb{R}$.

    Show that $\lim_{x \to c} f(x) = L$ if and only if $\lim_{x \to 0} f(x + c) = L$.

39. Show that, if $\|\cdot\|_1, \|\cdot\|_2$ are norms on $\mathbb{R}^d$ and $c_1, c_2 \in (0, \infty)$, then $c_1\|\cdot\|_1 + c_2\|\cdot\|_2$ is a norm.

40. Prove that every convergent sequence in $\mathbb{R}^d$ is bounded.

41. Recall that an open ball with respect to a norm $\|\cdot\|$ is defined by

    $$B_r(x) = \{y \in \mathbb{R}^n \; : \; \|x - y\| < r\},$$

    and a set $S \subset \mathbb{R}^n$ is open if for all $x \in S$, there exists $r > 0$ so that $B_r(x) \subset S$. Show that this definition of an open set does not depend on the norm used to define the open balls.

42. Give an open cover of $(0, 1)$ with no finite subcover. Also give a sequence in $(0, 1)$ without any subsequence that converges to a point in $(0, 1)$.

43. Say that a set $K \subset \mathbb{R}^d$ is *disconnected* if there exist sets A, B so that $K = A \cup B$, $A \cap B^c = \emptyset$ and $A^c \cap B = \emptyset$. Otherwise, it is connected. Show that $[0, 1]$ is connected while $(0, 1) \cap (1, 2)$ is disconnected.

44. Say that a set $K \subset \mathbb{R}^d$ is *path-connected* if for all $k_1, k_2 \in K$, there exists a continuous function $p\colon [0, 1] \to K$ such that $p(0) = k_1$ and $p(1) = k_2$. Let K be a compact, path-connected set and let $f\colon K \to \mathbb{R}$ be continuous on K. Prove that there exist $k_{\min}$ and $k_{\max}$ such that $f(K) = [f(k_{\min}), f(k_{\max})]$.

45. Show $\lim_{x \to c} x^3 = c^3$ for any $c \in \mathbb{R}$.

46. Use either the $\varepsilon - \delta$ definition of the limit or the Sequential Criterion for limits to establish the following limits:

    (a) $\lim_{x \to 2} \dfrac{1}{1-x} = -1$;

    (b) $\lim_{x \to 1} \dfrac{x}{1+x} = \dfrac{1}{2}$;

    (c) $\lim_{x \to 0} \dfrac{x^2}{|x|} = 0$, and

    (d) $\lim_{x \to 1} \dfrac{x^2 - x + 1}{x+1} = \dfrac{1}{2}$

47. Show that the following limits do not exist:

    (a) $\lim_{x \to 0} \dfrac{1}{x^2}$, with $x > 0$;

    (b) $\lim_{x \to 0} \dfrac{1}{\sqrt{x}}$, with $x > 0$;

    (c) $\lim_{x \to 0} (x + \operatorname{sgn}(x))$, and

    (d) $\lim_{x \to 0} \sin(1/x^2)$, with $x > 0$.

48. Let $c \in \mathbb{R}$ and let $f : \mathbb{R} \to \mathbb{R}$ be such that $\lim_{x \to c} (f(x))^2 = L$.

    Show that if $L = 0$, then $\lim_{x \to c} f(x) = 0$.

    Show that if $L \neq 0$, then f may not have a limit at c.

49. Let $f : \mathbb{R} \to \mathbb{R}$, let J be a closed interval in $\mathbb{R}$ and let $c \in J$.

    If f_2 is the restriction of f to J, show that if f has a limit at c then f_2 has a limit at c. Show the converse is not necessarily true.

50. Determine the following limits and state which theorems are used in each case.

    (a) $\lim_{x \to 2} \sqrt{\dfrac{2x+1}{x+3}}$, $(x > 0)$;

    (b) $\lim_{x \to 2} \dfrac{x^2 - 4}{x - 2}$, $(x > 0)$;

    (c) $\lim_{x \to 0} \sqrt{\dfrac{(x+1)^2 - 1}{x}}$, $(x > 0)$, and

    (d) $\lim_{x \to 1} \dfrac{\sqrt{x} - 1}{x - 1}$, $(x > 0)$.

51. Give examples of functions f and g such that f and g do not have limits at point c, but both $f + g$ and fg have limits at c.

52. Determine whether the following limits exist in $\mathbb{R}$:

   (a) $\lim_{x\to 0} \sin\left(\frac{1}{x^2}\right)$, with $x \neq 0$;

   (b) $\lim_{x\to 0} x\sin\left(\frac{1}{x^2}\right)$, with $x \neq 0$;

   (c) $\lim_{x\to 0} \operatorname{sgn}\sin\left(\frac{1}{x}\right)$, with $x \neq 0$, and

   (d) $\lim_{x\to 0} \sqrt{x}\sin\left(\frac{1}{x^2}\right)$, with $x > 0$.

53. Let $f : \mathbb{R} \to \mathbb{R}$ be s.t. $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. Assume $\lim_{x\to 0} f(x) = L$ exists. Prove that $L = 0$ and that f has a limit at every point $c \in \mathbb{R}$.

54. Let $K > 0$ and let $f : \mathbb{R} \to \mathbb{R}$ satisfy the condition

$$|f(x) - f(y)| \leq K|x - y|$$

   for all $x, y \in \mathbb{R}$. Show that f is continuous on $\mathbb{R}$.

55. Let $f : (0, 1) \to \mathbb{R}$ be bounded and s.t. $\lim_{x\to 0} f(x)$ does not exist.

   Show that there are two convergent sequences $(x_n), (y_n) \subseteq (0, 1)$ with $x_n, y_n \to 0$ and $f(x_n) \to \xi, f(y_n) \to \zeta$, but $\xi \neq \zeta$.

56. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous on $\mathbb{R}$ and let $P = \{x \in \mathbb{R} : f(x) > 0\}$. If $c \in P$, show that there exists a neighbourhood $V_\delta(c) \subseteq P$.

57. Prove that if an additive function is continuous at some point $c \in \mathbb{R}$, it is continuous on $\mathbb{R}$.

58. If f is a continuous additive function on $\mathbb{R}$, show that $f(x) = cx$ for all $x \in \mathbb{R}$, where $c = f(1)$.

59. Let $I = [a, b]$ and $f : I \to \mathbb{R}$ be a continuous function on I s.t. $\forall x \in I$, $\exists y \in I$ s.t. $|f(y)| \leq \frac{1}{2}|f(x)|$. Show $\exists c \in I$ s.t. $f(c) = 0$.

60. Show that every polynomial with odd degree has at least one real root.

61. Let $f : [0, 1] \to \mathbb{R}$ be continuous and s.t. $f(0) = f(1)$. Show $\exists c \in [0, \frac{1}{2}]$ s.t. $f(c) = f(c + \frac{1}{2})$.

62. Show that $f(x) = \frac{1}{x^2}$ is uniformly continuous on $A = [1, \infty)$, but not on $B = (0, \infty)$.

63. If $f(x) = x$ and $g(x) = \sin x$, show that f and g are both uniformly continuous on $\mathbb{R}$ but that their product is not uniformly continuous on $\mathbb{R}$.

64. Let $A \subseteq \mathbb{R}$ and suppose that f has the following property:

   $\forall \varepsilon > 0$, $\exists g_\varepsilon : A \to \mathbb{R}$ s.t. g_ε is uniformly continuous on A with $|f(x) - g_\varepsilon(x)| < \varepsilon$ for all $x \in A$.

   Show f is uniformly continuous on A.

65. Prove that a continuous $p-$periodic fonction on $\mathbb{R}$ is bounded and uniformly continuous on $\mathbb{R}$.

66. Use the definition to find the derivative of the function defined by $g(x) = \frac{1}{x}$, $x \in \mathbb{R}$, $x \neq 0$.

67. Prove that the derivative of an even differentiable function is odd, and vice-versa.

68. Let $a > b > 0$ and $n \in \mathbb{N}$ with $n \geq 2$.

    Show that $a^{1/n} - b^{1/n} < (a-b)^{1/n}$.

69. Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). Show that if $\lim_{x \to a} f'(x) = A$, then $f'(a)$ exists and equals A.

70. If $x > 0$, show $1 + \frac{1}{2}x - \frac{1}{8}x^2 \leq \sqrt{1+x} \leq 1 + \frac{1}{2}x$.

71. Show directly that the function defined by $h(x) = x^2$ is Riemann-integrable over $[a,b]$, $b > a \geq 0$. Furthermore show that $\int_a^b h = \frac{b^3 - a^3}{3}$.

72. Prove that $\int_0^1 g = \frac{1}{2}$ if

$$g(x) = \begin{cases} 1 & x \in (\frac{1}{2}, 1] \\ 0 & x \in [0, \frac{1}{2}] \end{cases}.$$

    Is that still true if $g(\frac{1}{2}) = 7$ instead?

73. Let $f : [a,b] \to \mathbb{R}$ be bounded and s.t. $f(x) \geq 0$ for all $x \in [a,b]$.

    Show $L(f) \geq 0$.

74. Let $f : [a,b] \to \mathbb{R}$ be increasing on $[a,b]$. If P_n partitions $[a,b]$ into n equal parts, show that

$$0 \leq U(P_n; f) - \int_a^b f \leq \frac{f(b) - f(a)}{n}(b-a).$$

75. Let $f : [a,b] \to \mathbb{R}$ be an integrable function and let $\varepsilon > 0$.

    If P_ε is the partition whose existence is asserted by the Riemann Criterion, show that $U(P; f) - L(P; f) < \varepsilon$ for all refinement P of P_ε.

76. Let $a > 0$ and $J = [-a, a]$. Let $f : J \to \mathbb{R}$ be bounded and let $\mathcal{P}^*$ be the set of all partitions P of J that contain 0 and are symmetric.

    Show $L(f) = \sup\{L(P; f) : P \in \mathcal{P}^*\}$.

77. Let J be as in the previous question and let f be integrable on J. If f is even (i.e. $f(-x) = f(x)$ for all x), show that

$$\int_{-a}^a f = 2\int_0^a f.$$

    If f is odd (i.e. $f(-x) = -f(x)$ for all x), show that

$$\int_{-a}^a f = 0.$$

78. Give an example of a function $f : [0,1] \to \mathbb{R}$ that is not integrable on $[0,1]$, but s.t. $|f|$ is integrable on $[0,1]$.

79. Let $f : [a,b] \to \mathbb{R}$ be integrable on $[a,b]$. Show $|f|$ is integrable on $[a,b]$ directly (without using a result seen in class).

80. If f is integrable on $[a,b]$ and
$$0 \le m \le f(x) \le M$$
for all $x \in [a,b]$, show that
$$m \le \left[\frac{1}{b-a}\int_a^b f^2\right]^{1/2} \le M.$$

81. If f is continuous on $[a,b]$ and $f(x) \ge 0$ for all $x \in [a,b]$, show there exists $c \in [a,b]$ s.t.
$$f(c) = \left[\frac{1}{b-a}\int_a^b f^2\right]^{1/2}.$$

82. If f is continuous on $[a,b]$ and $f(x) > 0$ for all $x \in [a,b]$, show that $\frac{1}{f}$ is integrable on $[a,b]$.

83. Let f be continuous on $[a,b]$. Define $H : [a,b] \to \mathbb{R}$ by
$$H(x) = \int_x^b f \quad \text{for all } x \in [a,b].$$
Find $H'(x)$ for all $x \in [a,b]$.

84. Suppose $f : [0,\infty) \to \mathbb{R}$ is continuous and $f(x) \ne 0$ for all $x > 0$. If
$$(f(x))^2 = 2\int_0^x f \quad \text{for all } x > 0,$$
show that $f(x) = x$ for all $x \ge 0$.

85. Let $f, g : [a,b] \to \mathbb{R}$ be continuous and s.t.
$$\int_a^b f = \int_a^b g.$$
Show that there exists $c \in [a,b]$ s.t. $f(c) = g(c)$.

86. Let $f : [0,3] \to \mathbb{R}$ be defined by
$$f(x) = \begin{cases} x & x \in [0,1) \\ 1 & x \in [1,2) \\ x & x \in [2,3] \end{cases}.$$
Find $F : [0,3] \to \mathbb{R}$, where
$$F(x) = \int_0^x f.$$
Where is F differentiable? What is F' there?

87. If $f : [0,1] \to \mathbb{R}$ is continuous and $\int_0^x f = \int_x^1 f$ for all $x \in [0,1]$, show that $f \equiv 0$.

88. Show that $\lim_{n\to\infty} \dfrac{nx}{1+n^2x^2} = 0$ for all $x \in \mathbb{R}$.

89. Show that if $f_n(x) = x + \frac{1}{n}$ and $f(x) = x$ for all $x \in \mathbb{R}$, $n \in \mathbb{N}$, then $f_n \rightrightarrows f$ on $\mathbb{R}$ but $f_n^2 \not\rightrightarrows g$ on $\mathbb{R}$ for any function g.

90. Let $f_n(x) = \frac{1}{(1+x)^n}$ for $x \in [0,1]$. Denote by f the pointwise limit of f_n on $[0,1]$. Does $f_n \rightrightarrows f$ on $[0,1]$?

91. Let (f_n) be the sequence of functions defined by $f_n(x) = \frac{x^n}{n}$, for $x \in [0,1]$ and $n \in \mathbb{N}$.

    Show that (f_n) converges uniformly to a differentiable function $f : [0,1] \to \mathbb{R}$, and that the sequence (f_n') converges pointwise to a function $g : [0,1] \to \mathbb{R}$, but that $g(1) \neq f'(1)$.

92. Show that $\lim\limits_{n\to\infty} \int_1^2 e^{-nx^2}\, dx = 0$.

93. Find the values of p for which the series $\sum\limits_{n=1}^{\infty} \frac{1}{n^p}$ converges.

94. Find the values of x for which the following series converge:

    (a) $\sum\limits_{n=1}^{\infty} (nx)^n$;

    (b) $\sum\limits_{n=1}^{\infty} x^n$;

    (c) $\sum\limits_{n=1}^{\infty} \frac{x^n}{n}$;

    (d) $\sum\limits_{n=1}^{\infty} \frac{x^n}{n^2}$, and

    (e) $\sum\limits_{n=1}^{\infty} \frac{x^n}{n!}$.

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96. Series question placeholder

97. Series question placeholder

98. Series question placeholder

99. Series question placeholder

100. Series question placeholder