MAT 2125 – Homework 1 – Solutions

(due at midnight on January 25, in Brightspace)

1 Practicing LaTeX

1. We could re-write the English sentence "For every real number, there exists a bigger real number" as $\forall x \in \mathbb{R}, \exists\, y \in \mathbb{R} : y > x$. Do a similar translation of the English sentence "For every integer, there exists a smaller integer."

2. Compute $\int_0^{10} \sin(x)\cos(x)dx$, showing at least two intermediate steps. Using the "align" environment or otherwise, vertically align the "=" sign between steps.

Solution: For both of the following, take a look at the provided .tex source file if you ran into difficulties.

1. We write:

$$\forall x \in \mathbb{N}\, \exists y \in \mathbb{N} : y < x.$$

2. We compute:

$$\begin{aligned} \int_0^{10} \sin(x)\cos(x)dx &= \frac{1}{2}\int_0^{10} \sin(2x)dx \\ &= -\frac{1}{4}\cos(2x)\Big|_0^{10} \\ &\approx -\frac{1}{4}(-0.839 - 1) \\ &\approx 0.460. \end{aligned}$$

2 Cardinality

Show that there exists a bijection between $\mathbb{Z}$ and $\mathbb{Q}$. **Hint:** If you can find a surjection in both directions, then you have shown that a bijection exists. This might be easier.

Proof: Write $\mathbb{Q} = \{\frac{m}{n} : m, n \in \mathbb{Z}, n > 0, \mathrm{GCD}(m,n) = 1\}$, where $\mathrm{GCD}(m,n)$ is the greatest common divisor of m, n. Define the map $f : \mathbb{Q} \to \mathbb{Z}$ by $f(\frac{m}{n}) = m$. To see that this is surjective, note that for all $m \in \mathbb{Z}$, $\frac{m}{1} \in \mathbb{Q}$ and $f(\frac{m}{1}) = m$.

Next, we define the map $g : \mathbb{Z} \mapsto \mathbb{Q}$ according to three cases:

1. For numbers of the form $2^a 3^b$ with $a, b \in \{0, 1, 2, \ldots\}$, set $g(2^a 3^b) = \frac{a}{b}$.
2. For numbers of the form $-2^a 3^b$ with $a, b \in \{0, 1, 2, \ldots\}$, set $g(-2^a 3^b) = -\frac{a}{b}$.
3. For all other numbers n, set $g(n) = 0$.

We need to check that g is well-defined, and then that it is surjective. To see that it is well-defined, we note that integers have unique prime decompositions, and $2, 3$ are prime. This means that every number can have *at most* one decomposition of the form $\pm 2^a 3^b$, so every number is in *at most* one case. It is also clear that every number n must be in *at least* one case. Thus, every number belongs to *exactly one case*, so it is well-defined.

To check that g is surjective, we consider some $\frac{m}{n} \in \mathbb{Q}$ and again consider three cases:

1. $\frac{m}{n} > 0$: $g(2^m 3^n) = \frac{m}{n}$.
2. $\frac{m}{n} < 0$: $g(-2^m 3^n) = \frac{m}{n}$.
3. $\frac{m}{n} = 0$: $g(5) = \frac{m}{n}$. $\blacksquare$

3 Calculations with Axioms

1. *Using only the field axioms,* show that the multiplicative identiy is unique. That is, show that if a, b are *both* multiplicative identities, then in fact $a = b$.

   Proof: Let a, b be two multiplicative identities in a field. Since a is a multiplicative identity,
   $$ab = b.$$
   Since b is a multiplicative identity,
   $$ab = a.$$
   Combining these two equations,
   $$b = ab = a.$$
   This completes the proof. ∎

2. *Using only the field axioms,* show that $(2x-1)(2x+1) = 4x^2 - 1$. **Note:** The field axioms don't define 2 or 4 are. Please take these to be shorthands for $2 = 1 + 1$ and $4 = 1 + 1 + 1 + 1$.

   Proof: Each equality is labeled with the field axiom used, as in the course notes:
   $$\begin{aligned}(2x-1)(2x+1) &\overset{\text{AM1}}{=} 2x(2x+1) + (-1)(2x+1)\\ &\overset{\text{AM1}}{=} (2x)(2x) + (1)2x + (-1)(2x) + (-1)(1)\\ &\overset{\text{AM1}}{=} (2x)(2x) + (1 + (-1))2x + (-1)(1)\\ &\overset{\text{A4}}{=} (2x)(2x) + (-1)(1)\\ &\overset{\text{A3}}{=} (2x)(2x) - 1\\ &\overset{\text{M1}}{=} ((2)(2))(x^2) - 1\\ &= ((1+1)(1+1))(x^2) - 1\\ &\overset{\text{AM1}}{=} (1(1+1) + 1(1+1))x^2 - 1\\ &\overset{\text{M3}}{=} 4x^2 - 1.\end{aligned}$$
   This completes the proof. ∎

3. *Using only the order axioms, usual arithmetic manipulations, and inequalities between concrete numbers,* prove the following: If $x \in \mathbb{R}$ satisfies $x < \epsilon$ for all $\epsilon > 0$, *then* $x \leq 0$. **Note:** The order axioms in the notes don't give concrete inequalities such as e.g. $1 > 0$, but we will show some of these in videos, class or DGD. For the purposes of this question you can take obvious inequalities between *specific integers* as given. That is, you could take $3 > 1$ as given, but should justify $x < 2x$.

   Proof: Assume first that $x > 0$. By O4 (and the fact that $0 < \frac{1}{2} < 1$), this means
   $$\left(\frac{1}{2}\right) x > \left(\frac{1}{2}\right) \cdot 0 = 0$$
   as well. By O3, since $\frac{x}{2} > 0$,
   $$\frac{x}{2} < \frac{x}{2} + \frac{x}{2} = x.$$
   Putting together these two sequences of inequalities, we have
   $$0 < \frac{x}{2} < x.$$
   But then we have found some number $\epsilon = \frac{x}{2} > 0$ so that $x > \epsilon$; this contradicts the original assumption. Thus, we conclude that our original assumption $x > 0$ is false; by O1, we conclude $x \leq 0$. ∎

4. Show that there exists some $x \in \mathbb{R}$ satisfying $x^2 + x = 5$. **Hint:** Find an interval $[a, b]$ for which $a^2 + a < 5$ and $b^2 + b > 5$, then try to adjust the proof that $\sqrt{2}$ exists.

   Proof: Let's follow the hint. Consider the interval $I = [0, 10]$, define $S = \{x \in I : x^2 + x < 5\}$, and define $A = \sup S$. Note that for $x \in [0, 1]$, $x^2 + x - 5 \leq 1^2 + 1 - 5 = -3 < 0$, so $A \geq 1$. Similarly, for $x \in [9, 10]$, $x^2 + x - 5 \geq 9^2 + 9 - 5 > 0$, so $A \leq 9$.

   Claim: $A^2 + A = 5$. This is shown in two parts: first we show that $A^2 + A \leq 5$, then we show that $A^2 + A \geq 5$.

   We show that $A^2 + A \leq 5$ by contradiction. Let's assume $A^2 + A > 5$. Then, by an earlier part of this homework, there exists some $0 < \epsilon < 1$ so that $A^2 + A > 5 + \epsilon$. But then for all $0 < \delta < \frac{\epsilon}{100}$, we have

$$
\begin{aligned}
(A - \delta)^2 + (A - \delta) &= A^2 - 2A\delta + \delta^2 + A - \delta \\
&\geq A^2 - (2)(10)(\delta) + A - \delta \\
&\geq A^2 + A - 21\delta \\
&> A^2 + A - \epsilon \\
&> 5.
\end{aligned}
$$

   Furthermore, since $A \geq 1$ and $\delta \leq 0.01$, we know that $A - \delta \in I$. Thus, in this case $A - \frac{\epsilon}{100} < A$ is also an upper bound on S, contradicting the fact that A is defined to be the least upper bound on S. We conclude that $A^2 + A \leq 5$.

   Next, we show that $A^2 + A \geq 5$ by contradiction. Let's assume $A^2 + A < 5$. Then, by an earlier part of this homework, there exists some $0 < \epsilon < 1$ so that $A^2 + A < 5 - \epsilon$. But then for all $0 < \delta < \frac{\epsilon}{100}$, we have

$$
\begin{aligned}
(A + \delta)^2 + (A + \delta) &= A^2 + A + (2A + 1 + \delta)\delta \\
&\leq A^2 + A + 22\delta \\
&< A^2 + A - \epsilon \\
&> 5.
\end{aligned}
$$

   Furthermore, since $A \leq 9$ and $\delta \leq 0.01$, we know that $A + \delta \in I$. Thus, in this case $A + \frac{\epsilon}{100} \in S$ and $A + \frac{\epsilon}{100} > A$, contradicting the fact that A is defined to be an upper bound on S. We conclude that $A^2 + A \leq 5$.

   Having shown $A^2 + A \leq 5$ and $A^2 + A \geq 5$, we conclude that $A^2 + A = 5$. ■

5. Let $A, B \subset \mathbb{R}$ and define $C = \{x - y : x \in A, y \in B\}$. Prove that $\inf(C) = \inf(A) - \sup(B)$.

   Proof: We prove the statement in two parts: first we show that $\inf(C) \leq \inf(A) - \sup(B)$, then we show that $\inf(C) \geq \inf(A) - \sup(B)$.

   To prove the first part, fix $\epsilon > 0$. Then there exists some $a \in A$ and $b \in B$ so that $a \leq \inf(A) + \frac{\epsilon}{2}$, $b \geq \sup(B) - \frac{\epsilon}{2}$. Thus, $a - b \in C$ and $a - b \leq \inf(A) - \sup(B) + \epsilon$. Thus, $\inf(C) \leq \inf(A) - \sup(B) + \epsilon$. Applying the previous question in the homework, we conclude that $\inf(C) \leq \inf(A) - \sup(B)$.

   To prove the second part, consider $c \in C$. Then there exists $a \in A, b \in B$ so that $c = a - b$. By the definition of the infimum and supremum, $a \geq \inf(A)$ and $b \leq \sup(B)$, so $c = a - b \geq \inf(A) - \sup(B)$. We conclude that $\inf(C) \geq \inf(A) - \sup(B)$.

   Since we have shown $\inf(C) \leq \inf(A) - \sup(B)$ and $\inf(C) \geq \inf(A) - \sup(B)$, we conclude by the first order axiom $\inf(C) = \inf(A) - \sup(B)$. ■

6. Consider a set S with $0 \leq \sup(S) = A < \infty$ and $A \notin S$. Show that for all $\epsilon > 0$, $S \cap [A-\epsilon, A]$ is nonempty. Using this fact or otherwise, conclude that in fact $S \cap [A-\epsilon, A]$ is infinite.

   Proof: We prove the first claim by contradiction. Assume there exists some $\epsilon > 0$ so that $S \cap [A-\epsilon, A]$ is empty. Since A is an upper bound for S, we also know that $S \cap (A, \infty)$ is empty. Thus, $S \cap [A-\epsilon, \infty)$ is empty. But this means that $A - \epsilon < A$ is an upper bound for s, contradicting the fact that A is the least upper bound for S. We conclude that in fact $S \cap [A-\epsilon, A]$ is not empty.

   We also prove the second part by contradiction. Assume there exists some $\epsilon > 0$ so that $S \cap [A-\epsilon, A]$ is empty. Then we can enumerate its elements, $\{b_1, \ldots, b_n\}$. Let $B = \max(b_1, \ldots, b_n\}$. Since $A \notin S$, we know that $b_1, \ldots, b_n < A$. Since B is a maximum of finitely many elements, this means that $B < A$ as well. But then $A > A - \frac{A-B}{2} > B$, so $[A - \frac{A-B}{2}, A] \cap S$ is empty. But this is impossible, by the first part of the question. This completes the proof. ■

4 Induction

Somebody walks up to you with a proof by induction of the statement "*For any integer $N \in \mathbb{N}$, all collections of N sheep are the same colour*," as follows:

- **Notation:** Let $x_1, x_2, \ldots,$ be the colours of all sheep in the world, put in some order.
- **Base Case:** Obviously the first sheep is a single colour, x_1.
- **Inductive Case:** Assume that the statement is true up to some integer n. By the inductive assumption, the collection of the first n sheep $\{x_1, \ldots, x_n\}$ are one colour (label this "colour 1"), and the collection of the last n sheep $\{x_2, \ldots, x_{n+1}\}$ are also one colour (label this "colour 2" - note that we haven't yet shown it is the same colour as the first collection). Since $\{x_2, \ldots, x_n\}$ are in *both* sets, we must have that "colour 1" and "colour 2" are the same, and so $\{x_1, \ldots, x_{n+1}\}$ are all one colour.

Explain why this purported proof fails by identifying and explaining a (significant) false statement. **Note:** we are asking for an *important, actually-false* statement, not *merely* something like a typo or insufficiently-formal justification for an assertion.

Solution: The critical error is in the following part of the argument, in the case $n = 1$:

> "the collection of the first n sheep $\{x_1, \ldots, x_n\}$ are one colour, and the collection of the last n sheep $\{x_2, \ldots, x_{n+1}\}$ are also one (possibly different) colour. Since $\{x_2, \ldots, x_n\}$ are in *both* sets, both sets must in fact be the same colour, and so $\{x_1, \ldots, x_{n+1}\}$ are all one colour."

Consider the case $n = 1$. Then the collection $\{x_2, \ldots, x_n\}$ is actually *empty*, and so we can't conclude that the two sets $\{x_1\}, \{x_2\}$ share any sheep, and so we can't conclude that they are the same colour. ■