MAT 2125 – Homework 1

(due at midnight on January 25, in Brightspace)

1 Practicing LaTeX

1. We could re-write the English sentence "For every real number, there exists a bigger real number" as $\forall x \in \mathbb{R}, \exists\, y \in \mathbb{R} : y > x$. Do a similar translation of the English sentence "For every integer, there exists a smaller integer."

2. Compute $\int_0^{10} \sin(x)\cos(x)dx$, showing at least two intermediate steps. Using the "align" environment or otherwise, vertically align the "=" sign between steps.

2 Cardinality

Show that there exists a bijection between $\mathbb{Z}$ and $\mathbb{Q}$. **Hint:** If you can find a surjection in both directions, then you have shown that a bijection exists. This might be easier.

3 Calculations with Axioms

1. *Using only the field axioms,* show that the multiplicative identiy is unique. That is, show that if a, b are *both* multiplicative identities, then in fact $a = b$.

2. *Using only the field axioms,* show that $(2x-1)(2x+1) = 4x^2 - 1$. **Note:** The field axioms don't define 2 or 4 are. Please take these to be shorthands for $2 = 1 + 1$ and $4 = 1 + 1 + 1 + 1$.

3. *Using only the order axioms, usual arithmetic manipulations, and inequalities between concrete numbers,* prove the following: If $x \in \mathbb{R}$ satisfies $x < \epsilon$ for all $\epsilon > 0$, *then* $x \leq 0$. **Note:** The order axioms in the notes don't give concrete inequalities such as e.g. $1 > 0$, but we will show some of these in videos, class or DGD. For the purposes of this question you can take obvious inequalities between *specific integers* as given. That is, you could take $3 > 1$ as given, but should justify $x < 2x$.

4. Show that there exists some $x \in \mathbb{R}$ satisfying $x^2 + x = 5$. **Hint:** Find an interval $[a, b]$ for which $a^2 + a < 5$ and $b^2 + b > 5$, then try to adjust the proof that $\sqrt{2}$ exists.

5. Let $A, B \subset \mathbb{R}$ and define $C = \{x - y \,:\, x \in A, y \in B\}$. Prove that $\inf(C) = \inf(A) - \sup(B)$.

6. Consider a set S with $0 \leq \sup(S) = A < \infty$ and $A \notin S$. Show that for all $\epsilon > 0$, $S \cap [A - \epsilon, A]$ is nonempty. Using this fact or otherwise, conclude that in fact $S \cap [A - \epsilon, A]$ is infinite.

4 Induction

Somebody walks up to you with a proof by induction of the statement "*For any integer $N \in \mathbb{N}$, all collections of N sheep are the same colour*," as follows:

- **Notation:** Let $x_1, x_2, \ldots,$ be the colours of all sheep in the world, put in some order.
- **Base Case:** Obviously the first sheep is a single colour, x_1.
- **Inductive Case:** Assume that the statement is true up to some integer n. By the inductive assumption, the collection of the first n sheep $\{x_1, \ldots, x_n\}$ are one colour (label this "colour 1"), and the collection of the last n sheep $\{x_2, \ldots, x_{n+1}\}$ are also one colour (label this "colour 2" - note that we haven't yet shown it is the same colour as the first collection). Since $\{x_2, \ldots, x_n\}$ are in *both* sets, we must have that "colour 1" and "colour 2" are the same, and so $\{x_1, \ldots, x_{n+1}\}$ are all one colour.

Explain why this purported proof fails by identifying and explaining a (significant) false statement. **Note:** we are asking for an *important, actually-false* statement, not *merely* something like a typo or insufficiently-formal justification for an assertion.