MAT 2125 – Homework 2

(due at midnight on February 12, in Brightspace)

1 Formality

In the first assignment, we asked you to justify many steps directly from the axioms for $\mathbb{R}$. Starting with this homework, we can be slightly less formal – we can take it that it is understood what the number $\frac{21}{4}$ means for instance, as is the case for more exotic numbers like $3^{\frac{1}{7}}$. We can also freely use "obvious" facts, like the fact that the maximum of a finite collection of real numbers exists and is finite, without directly citing anything.

With this comes some danger. As we will see in Section 4 of this homework, sometimes "obvious"-looking statements are actually false. The only way around this is to use our judgement. Since this is an introductory course, we ask you to err on the side of caution: if you haven't seen something proved, and it looks like the sort of thing that could be an exercise, you should prove it.

2 Limits

1. Suppose that $\{a_n\}$ is a bounded sequence and $\lim_{n\to\infty} b_n = 0$. Show that $\lim_{n\to\infty} a_n b_n = 0$.
2. Suppose that $\{a_n\}$ is a sequence and $\lim_{n\to\infty} a_n$ exists. Show that $\{a_n\}$ is a bounded sequence.
3. Show that, for all $c \in (0, \infty)$, $\lim_{n\to\infty} c^{\frac{1}{n}} = 1$. **Vague hint:** For $c > 1$, consider the sequence x_n defined by $x_n = (1 + a_n)^n$, find a good linear approximation of $(1 + a_n)^n$, and apply an earlier part of the this question.
4. Consider the sequence given by the recursion $a_{n+1} = \frac{1}{2}(a_n + a_n^{-1})$, with some initial condition $a_1 \in (-\infty, 0) \cup (0, \infty)$. Find and prove the limit (if it exists) for initial conditions $a_1 = 3$, $a_1 = 0.1$.

3 Subsequences

Let $\{a_n\}$ be a sequence with no convergent subsequences. Show that $|a_n| \to \infty$.

4 Limit Inferior and Limit Superior

We define the **limit inferior** and the **limit superior** of a sequence as follows:

$$\liminf_{n\to\infty} a_n = \lim_{n\to\infty} \inf\{a_k \mid k \geq n\}$$
$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} \sup\{a_k \mid k \geq n\}.$$

1. Let $\{a_n\}$ be a bounded sequence. Show that $\liminf_{n\to\infty} a_n$ and $\limsup_{n\to\infty} a_n$ exist and are in $\mathbb{R}$.
2. Let $\{a_n\}$ be an unbounded sequence. Show that either $\liminf_{n\to\infty} a_n = -\infty$ or $\limsup_{n\to\infty} a_n = \infty$ (or possibly both).
3. Let $\{a_n\}$, $\{b_n\}$ be two sequences. Show that

$$\liminf_{n\to\infty} a_n + \limsup_{n\to\infty} b_n \leq \limsup_{n\to\infty} (a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n.$$

   Furthermore, find a pair of sequences for which the second inequality is strict.