MAT 2125
Elementary Real Analysis

Exercises – Solutions – Q2-Q4

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2. Let $c > 0$ be a real number.

   (a) If $c > 1$, show that $c^n \geq c$ for all $n \in \mathbb{N}$ and that $c^n > 1$ if $n > 1$.
   (b) If $0 < c < 1$, show that $c^n \leq c$ for all $n \in \mathbb{N}$ and that $c^n < 1$ if $n > 1$.

Proof. The statement is clearly not true if $n = 0$: as a result, we must interpret $\mathbb{N}$ to stand for the set $\mathbb{N} = \{1, 2, 3, \ldots\}$, without the 0. Generally, we use whatever "version" of $\mathbb{N}$ is appropriate.

(a) If $c > 1$, $\exists x \in \mathbb{R}$ such that $x > 0$ and $c = 1 + x$. Let $n \in \mathbb{N}$. First note that $n - 1 \geq 0$ and so $(n - 1)x > 0$.

Then, by Bernoulli's Inequality,

$$c^n = (1 + x)^n \geq 1 + nx = 1 + x + (n - 1)x \geq 1 + x = c.$$

Furthermore, $n - 1 > 0$ and $(n - 1)x > 0$ if $n > 1$.

In that case, the last inequality above is strict and so $c^n > c > 1$, which implies $c^n > 1$ by transitivity of $>$.

(b) If $0 < c < 1$, there exists $b > 1$ such that $c = \frac{1}{b}$. Indeed, $\frac{1}{c}$ is such that $c \cdot \frac{1}{c} = 1$. As $c > 0$, then $\frac{1}{c} > 0$ since the product $c \cdot \frac{1}{c} = 1$ is positive.

But $c < 1$, so that $1 = c \cdot \frac{1}{c} < \frac{1}{c}$.

In particular, if we let $b = \frac{1}{c}$, then $b > 1$ and so we can apply part (a) of this question to get $b^n \geq b$ for all $n \in \mathbb{N}$ and $b^n > 1$ if $n > 1$.

Let $n \in \mathbb{N}$. Then

$$\frac{1}{c^n} = b^n \geq b = \frac{1}{c}$$

so that $c \geq c^n$ and

$$\frac{1}{c^n} = b^n > 1$$

so that $1 > c^n$ if $n > 1$. ■

3. Let $c > 0$ be a real number.

   (a) If $c > 1$ and $m, n \in \mathbb{N}$, show that $c^m > c^n$ if and only if $m > n$.
   (b) If $0 < c < 1$ and $m, n \in \mathbb{N}$, show that $c^m > c^n$ if and only if $m < n$.

Proof.

(a) It is sufficient to show that if $m \geq n$, then $c^m \geq c^n$. (Why is this the case? Don't let this slip by without understanding.)

If $m = n$, the result is clear. So we consider $m > n$.

In this case, $\exists k \geq 1$ such that $m = n + k$. An easy induction exercise shows that $c^{n+k} = c^n c^k$ for for all integers n and k (from this point on, we will assume and apply freely all the usual techniques of algebra).

In particular, using the previous problem,

$$c^m = c^{n+k} = c^n c^k \geq c^n \cdot c > c^n \cdot 1 = c^n$$

and so $c^m > c^n$.

(b) This can be shown from part (a) using the technique from the previous question. ■

4. Let $S_2 = \{x \in \mathbb{R} \mid x > 0\}$. Does S_2 have lower bounds? Does S_2 have upper bounds? Does $\inf S_2$ exist? Does $\sup S_2$ exist? Prove your statements.

Proof.

Does S_2 have lower bounds? Yes.
By definition, any negative real number is a lower bound (so is 0).

Does S_2 have upper bounds? No.
Assume that it does. By the completeness of $\mathbb{R}$, $\alpha = \sup \mathbb{R}$ exists. In particular, $\alpha \geq n$ for all $n \in \mathbb{N}$, which contradicts the Archimedean Property of $\mathbb{R}$. Hence S_2 has no upper bound.

Does $\inf S_2$ exist? Yes.
Consider the set $-S_2 = \{x \in \mathbb{R} \mid -x \in S_2\} = \{x \in \mathbb{R} \mid x < 0\}$. By construction, 0 is an upper bound of $-S_2$. Note furthermore that neither S_2 nor $-S_2$ are empty.

By completeness of $\mathbb{R}$, $\sup(-S_2)$ exists. Right?

One definition of completeness is that any non-empty bounded subset of $\mathbb{R}$ has a supremum. But $-S_2$ is only bounded above, not below. How can we conclude that $\sup(-S_2)$ exists?

That definition is one particular version of the Completeness Property of $\mathbb{R}$. An **equivalent** way of stating it is: *The ordered set F is **complete** if for any $\varnothing \neq S \subset F$, S has a supremum in F whenever S is bounded above and an infimum in F whenever S is bounded below.*

But $\sup(-S_2) = -\inf S_2$. Indeed, let $u = \sup(-S_2)$. Then $u \geq -x$ for all $-x \in -S_2$ and if v is another upper bound of $-S_2$ then $u \leq v$.

Note that if v is an upper bound of $-S_2$, then $v \geq -x$ for all $-x \in -S_2$, **i.e.** $-v \leq x$ for all $x \in S_2$: as a result, $-v$ is a lower bound of S_2.

Similarly, if $-v$ is a lower bound of S_2, v is automatically an upper bound of $-S_2$. Then any lower bound of S_2 is of the form $-v$, where v is an upper bound of $-S_2$.

Then, $-u \leq x$ for all $x \in S_2$ and $-v \leq -u$ whenever $-v$ is a lower bound of S_2. Hence $-u = \inf S_2$ and so $u = -\inf S_2$.

As $\sup(-S_2) = -\inf S_2$ exists, so does $\inf S_2$.

Does $\sup S_2$ exist? No.
See second item. ■