MAT 2125
Elementary Real Analysis

Exercises – Solutions – Q5-Q8

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5. Let $S_4 = \left\{1 - \frac{(-1)^n}{n} \mid n \in \mathbb{N}\right\}$. Find $\inf S_4$ and $\sup S_4$.

Solution. The first few elements of S_4 are

$$2, \frac{1}{2}, \frac{4}{3}, \frac{3}{4}, \frac{6}{5}, \frac{5}{6}, \dots .$$

This gives us the idea that S_4 is bounded above by 2 and below by $\frac{1}{2}$. To show that this is indeed the case, note that $(-1)^n$ only takes on the values -1 and 1, whatever the integer n.

Technically, this also has to be shown. One proceeds by induction.

The **base case** is clear: when $n = 1$, $(-1)^1 = -1 \in \{1, -1\}$.

Now, on to the **induction step**: suppose $(-1)^k \in \{1, -1\}$.

Then

$$(-1)^{k+1} = (-1)^k(-1) = \begin{cases} 1(-1) = -1 \\ (-1)(-1) = 1 \end{cases}.$$

Hence $(-1)^{k+1} \in \{1, -1\}$.

By induction, $(-1)^n \in \{-1, 1\}$ for all $n \in \mathbb{N}$.

Thus $-1 \leq (-1)^n \leq 1$ for all $n \geq 1$. (In practice, we need only show it once and refer to the result if we need it in the future.)

For any $n \geq 2$, we then have $-n \leq -1 \leq (-1)^n$ and $\frac{n}{2} \geq 1 \geq (-1)^n$, that is

$$-n \leq (-1)^n \leq \frac{n}{2}.$$

A quick check shows the inequalities also hold for $n = 1$.

Then, for $n \geq 1$,

$$-n \leq (-1)^n \leq \frac{n}{2}$$
$$\therefore -1 \leq \frac{(-1)^n}{n} \leq \frac{1}{2}$$
$$\therefore 1 \geq -\frac{(-1)^n}{n} \geq -\frac{1}{2}$$
$$\therefore 2 \geq 1 - \frac{(-1)^n}{n} \geq \frac{1}{2}.$$

Hence $2 \geq s \geq \frac{1}{2}$ for all $s \in S_4$, *i.e.* 2 is an upper bound and $\frac{1}{2}$ is a lower bound of S_4.

By completeness of $\mathbb{R}$, S_4 possesses a supremum and an infimum in $\mathbb{R}$. If $u = \sup S_4 < 2$, there is a contradiction as $u \not\geq s$ for all $s \in S_4$ (it "misses" the element 2 in S_4).

Thus, $\sup S_4 \geq 2$. But 2 is already an upper bound so $\sup S_4 \leq 2$. Consequently $\sup S_4 = 2$. Similarly, $\inf S_4 = \frac{1}{2}$. ■

6. Let $S \subseteq \mathbb{R}$ be non-empty. Show that if $u = \sup S$ exists, then for every number $n \in \mathbb{N}$ the number $u - \frac{1}{n}$ is not an upper bound of S, but the number $u + \frac{1}{n}$ is.

Proof. Let $n \geq 1$. Then $\frac{1}{n} > 0$ and $u < u + \frac{1}{n}$. Since $s \leq u$ for all $s \in S$, $s < u + \frac{1}{n}$ for all $s \in S$ by transitivity of $<$. Consequently, $u + \frac{1}{n}$ is an upper bound of S.

Furthermore, $u - \frac{1}{n} < u$. Since u is the least upper bound, $u - \frac{1}{n}$ cannot be an upper bound (as it would then be lesser upper bound than u, a contradiction). This completes the proof. Or does it?

We haven't used the hypothesis $S \neq \varnothing$. Where does it fit?

The definition of an upper bound implies that every real number is an upper bound of the empty set. Indeed, if $v \in \mathbb{R}$, then $v \geq s$ for all $s \in \varnothing$ automatically as there is **no** $s \in \varnothing$.

The proof rests on the fact that $u = \sup S$. But $\sup \varnothing$ does not exist as we just discussed. OK. Now it's the end for real. $\blacksquare$

7. If $S = \left\{\frac{1}{n} - \frac{1}{m} \mid m, n \in \mathbb{N}\right\}$, find $\inf S$ and $\sup S$.

Solution. The set $S = \left\{ \frac{1}{n} - \frac{1}{m} \mid n, m \in \mathbb{N} \right\}$ is bounded above by 1 and below by -1 since

$$\frac{1}{n} \leq 1 \leq 1 + \frac{1}{m} \quad \text{and} \quad \frac{1}{m} \leq 1 \leq 1 + \frac{1}{n} \implies -1 \leq \frac{1}{n} - \frac{1}{m} \leq 1, \quad \forall m, n \in \mathbb{N}.$$

Note that S is not empty as $0 = \frac{1}{2} - \frac{1}{2}$ is in S, say.

By completeness of $\mathbb{R}$, S thus has a supremum and an infimum.

By definition, $s^* = \sup S \leq 1$. Suppose that $s^* < 1$. Then $\exists \varepsilon > 0$ such that $s^* = 1 - \varepsilon$. Furthermore,

$$\frac{1}{n} - \frac{1}{m} \leq 1 - \varepsilon, \quad \forall m, n \in \mathbb{N}.$$

In particular, if $n = 1$, then

$$1 - \frac{1}{m} \leq 1 - \varepsilon, \quad \forall m \in \mathbb{N}.$$

Equivalently, $\varepsilon \leq \frac{1}{m}$ for all integers m so that $\frac{1}{\varepsilon}$ is an upper bound for $\mathbb{N}$.

This contradicts the Archimedean Property of $\mathbb{R}$. Hence $s^* \not< 1$ and so $s^* = 1$.

To prove that $\inf S = -1$, proceed along the same lines. $\blacksquare$

8. Let X be a non-empty set and let $f : X \to \mathbb{R}$ have bounded range in $\mathbb{R}$. If $a \in \mathbb{R}$, show that

$$\sup\{a + f(x) : x \in X\} = a + \sup\{f(x) : x \in X\}$$
$$\inf\{a + f(x) : x \in X\} = a + \inf\{f(x) : x \in X\}.$$

Proof. Let $f(X) = \{f(x) \mid x \in X\}$. By hypothesis, $f(X)$ is bounded and not empty and so has a supremum in $\mathbb{R}$, say u^*.

We need to show $\sup\{a + f(x); x \in X\} = a + u^*$.

To do so, first note that $a+u^*$ is an upper bound of $\sup\{a+f(x) \mid x \in X\}$ since $u^* \geq f(x)$ for all $x \in X$; as a result $a + u^* \geq a + f(x)$ for all $x \in X$.

(By completeness of $\mathbb{R}$, this means that $\sup\{a + f(x) \mid x \in X\}$ does indeed have a supremum.)

Next, we need to show that $a + u^*$ is the smallest upper bound of $\{a + f(x) \mid x \in X\}$.

Suppose v is another upper bound of $\{a + f(x) \mid x \in X\}$. Then $v \geq a + f(x)$ for all $x \in X$; in particular, $v - a$ is an upper bound of $f(X)$.

By hypothesis, $v - a \geq u^*$, hence $v \geq a + u^*$. Consequently, $a + u^*$ is the least upper bound of $\{a + f(x) \mid x \in X\}$, i.e.

$$\sup\{a + f(x) \mid x \in X\} = a + u^*.$$

The proof for the other equality proceeds in a similar manner. $\blacksquare$