MAT 2125
Elementary Real Analysis

Exercises – Solutions – Q10-Q12

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10. Let X be a non-empty set and let $f, g : X \to \mathbb{R}$ have bounded range in $\mathbb{R}$. Show that

$$\sup\{f(x) + g(x) \mid x \in X\} \leq \sup\{f(x) \mid x \in X\} + \sup\{g(x) \mid x \in X\}$$

$$\inf\{f(x) \mid x \in X\} + \inf\{g(x) \mid x \in X\} \leq \inf\{f(x) + g(x) \mid x \in X\}.$$

Solution. Let $f(X) = \{f(x) \mid x \in X\}$ and $g(X) = \{g(x) \mid x \in X\}$. By hypothesis, $f(X)$ and $g(X)$ are both bounded and not empty, so they each have a supremum in $\mathbb{R}$, say u^* and v^* respectively.

Since $f(x) \leq u^*$ and $g(x) \leq v^*$ for all $x \in X$, then $f(x)+g(x) \leq u^*+v^*$ for all $x \in X$.

Hence $\{f(x) + g(x) \mid x \in X\}$ has a supremum in $\mathbb{R}$, as it is a bounded non-empty subset of $\mathbb{R}$. Let w^* be that supremum, i.e. the smallest upper bound of $\{f(x) + g(x) \mid x \in X\}$.

Since $u^* + v^*$ is also an upper bound of that set, it's automatically larger than w^*. Note that we can not in general say more: it is **not** true, in general, that $w^* = u^* + v^*$.

Indeed, take $X = [1, 2]$ and let f and g be defined by

$$f(x) = \frac{1}{x} \quad \text{and} \quad g(x) = -\frac{1}{x}, \quad \forall x \in X.$$

Then $f(X) = \{\frac{1}{x} \mid x \in X\}$, $g(X) = \{-\frac{1}{x} \mid x \in X\}$ and $u^* = 1$, $v^* = -\frac{1}{2}$ and $w^* = 0$ (you should show these results!), and $w^* \leq u^* + v^*$ but $w^* \neq u^* + v^*$.

(Compare this result with the one from the previous question; what is the difference?)

The other inequality is tackled in a similar manner. ■

11. Let X and Y be non-empty sets and let $h : X \times Y \to \mathbb{R}$ have bounded range in $\mathbb{R}$. Let $F : X \to \mathbb{R}$ and $G : Y \to \mathbb{R}$ be defined by

$$F(x) = \sup\{h(x,y) \mid y \in Y\} \quad \text{and} \quad G(y) = \sup\{h(x,y) \mid x \in X\}.$$

Show that

$$\begin{aligned} \sup\{h(x,y) \mid (x,y) \in X \times Y\} &= \sup\{F(x) \mid x \in X\} \\ &= \sup\{G(y) \mid y \in Y\}. \end{aligned}$$

Proof. Let $h(X,Y) = \{h(x,y) \mid (x,y) \in X \times Y\}$. By definition, $h(X,Y)$ is bounded and not empty, so it has a supremum in $\mathbb{R}$, and F and G are well-defined.

Let $\alpha = \sup h(X,Y)$. Then $\alpha \geq h(x,y)$ for all $x \in X$ and $y \in Y$. In particular, if $x \in X$ is fixed, $\alpha \geq h(x,y)$ for all $y \in Y$. But $F(x)$ is the smallest upper bound of $\{h(x,y) \mid y \in Y\}$, so $\alpha \geq F(x)$.

But x was arbitrary, so $\alpha \geq F(x)$ for all $x \in X$. Hence α is an upper bound of $\{F(x) \mid x \in X\}$; by completeness, $\{F(x) \mid x \in X\}$ has a supremum in $\mathbb{R}$, say β. Then $\alpha \geq \beta$, by definition of the supremum.

Again, fix $x \in X$. Then $\beta \geq F(x) \geq h(x,y)$ for all $y \in Y$. Hence, for any $x \in X$, $\beta \geq h(x,y)$ for all $y \in Y$. As a result, β is an upper bound of $h(X,Y)$. Then $\beta \geq \alpha$, by definition of the supremum.

Combining these two results yields $\alpha = \beta$ (now do the other). ■

12. Show there exists a positive real number u such that $u^2 = 3$.

Solution. We first show that u is not rational (even though that wasn't part of the question, it will be informative).

Suppose the equation $r^2 = 3$ has a positive root r in $\mathbb{Q}$. Let $r = p/q$ with $\gcd(p, q) = 1$ be that solution. Then $p^2/q^2 = 3$, or $p^2 = 3q^2$. Hence p^2 is a multiple of 3, and so p is also a multiple of 3.

(Indeed, if p is not a multiple of 3, then neither is p^2. Let $p = 3k + 1$ or $p = 3k + 2$. Then $p^2 = 3(3k^2 + 2k) + 1$ or $p^2 = 3(3k^2 + 4k + 1) + 1$, neither of which is a multiple of 3.)

Set $p = 3m$. Then $(3m)^2 = 3q^2$, which is the same as $3m^2 = q^2$. Then q^2 is a multiple of 3, and so q is also a multiple of 3.

Consequently, p and q are both divisible by 3, which contradicts the hypothesis $\gcd(p, q) = 1$. The equation $r^2 = 3$ cannot then have a solution in $\mathbb{Q}$.

But we haven't shown yet that the equation had a solution in $\mathbb{R}$.

Consider the set $S = \{s \in \mathbb{R}^+ : s^2 < 3\}$, where $\mathbb{R}^+$ denotes the set of positive real numbers.

This set in not empty as $1 \in S$. Furthermore, S is bounded above by 3. (Indeed, if $t \geq 3$, then $t^2 \geq 9 > 3$, whence $t \notin S$.)

By completeness of $\mathbb{R}$, $x = \sup S \geq 1$ exists. It will be enough to show that neither $x^2 < 3$ and $x^2 > 3$ can hold. The only remaining possibility will be that $x = \sqrt{3}$.

- If $x^2 < 3$, then $\frac{2x+1}{3-x^2} > 0$. By the Archimedean property, $\exists n > 0$ such that $\frac{2x+1}{3-x^2} < n$. By re-arranging the terms, we get

$$0 < \frac{1}{n}(2x+1) < 3 - x^2.$$

Then

$$\left(x+\frac{1}{n}\right)^2 = x^2+\frac{2x}{n}+\frac{1}{n^2} \quad \leq \quad x^2+\frac{2x}{n}+\frac{1}{n}$$
$$= \quad x^2+\frac{1}{n}(2x+1) < x^2+3-x^2 = 3.$$

Since $(x+\frac{1}{n})^2 < 3$, $x+\frac{1}{n} \in S$. But $x < x+\frac{1}{n}$; x is then not an upper bound of S, which contradicts the fact that $x = \sup S$. Thus $x^2 \not< 3$.

- If $x^2 > 3$, then $\frac{2x}{x^2-3} > 0$. By the Archimedean property, $\exists n > 0$ such that $\frac{2x}{x^2-3} < n$. By re-arranging the terms, we get

$$0 > -\frac{2x}{n} > 3 - x^2.$$

Then

$$\left(x-\frac{1}{n}\right)^2 = x^2 - \frac{2x}{n} + \frac{1}{n^2} > x^2 - \frac{2x}{n} > x^2 + 3 - x^2 = 3.$$

Since $(x-\frac{1}{n})^2 > 3$, $x-\frac{1}{n}$ is an upper bound of S. But $x > x-\frac{1}{n}$; x can not then be the supremum of S, which is a contradiction. Thus $x^2 \not> 3$.

That leaves only one alternative (since we know that $x \in \mathbb{R}$): $x^2 = 3$, whence $x = u = \sqrt{3} > 0$. ■