MAT 2125
Elementary Real Analysis

Exercises – Solutions – Q13-Q15

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10. Show there exists a positive real number u such that $u^3 = 2$.

Solution. Consider the set $S = \{s \in \mathbb{R}^+ : s^3 < 2\}$, where $\mathbb{R}^+$ denotes the set of positive real numbers.

This set in not empty as $1 \in S$. Furthermore, S is bounded above by 2. (Indeed, if $t \geq 2$, then $t^3 \geq 8 > 2$, whence $t \notin S$.)

By completeness of $\mathbb{R}$, $x = \sup S \geq 1$ exists. It will be enough to show that neither $x^3 < 2$ and $x^3 > 2$ can hold. The only remaining possibility will be that $x = \sqrt[3]{2}$.

- If $x^3 < 2$, then $\frac{3x^2+3x+1}{2-x^3} > 0$. By the Archimedean property, $\exists n > 0$ such that $\frac{3x^2+3x+1}{2-x^3} < n$. By re-arranging the terms, we get

$$0 < \frac{1}{n}(3x^2 + 3x + 1) < 2 - x^3.$$

Then

$$\left(x+\frac{1}{n}\right)^3 = x^3+\frac{3x^2}{n}+\frac{3x}{n^2}+\frac{1}{n^3} \leq x^3+\frac{3x^2}{n}+\frac{3x}{n}+\frac{1}{n}$$
$$= x^3+\frac{1}{n}(3x^2+3x+1) < x^3+2-x^3 = 2.$$

Since $(x+\frac{1}{n})^3 < 2$, $x+\frac{1}{n} \in S$. But $x < x+\frac{1}{n}$; x is then not an upper bound of S, which contradicts the fact that $x = \sup S$. Thus $x^3 \not< 2$.

- If $x^3 > 2$, then $\frac{3x^2+1}{x^3-2} > 0$. By the Archimedean property, $\exists n > 0$ such that $\frac{3x^2+1}{x^3-2} < n$. By re-arranging the terms, we get

$$0 > -\frac{(3x^2+1)}{n} > 2-x^3.$$

Then

$$\left(x-\frac{1}{n}\right)^3 = x^3 - \frac{3x^2}{n} + \frac{3x}{n^2} - \frac{1}{n^3} \geq x^3 - \frac{3x^2}{n} - \frac{1}{n^3} \geq x^3 - \frac{3x^2}{n} - \frac{1}{n}$$
$$= x^3 - \frac{1}{n}(3x^2+1) > x^3 + 2 - x^3 = 2.$$

Since $(x-\frac{1}{n})^3 > 2$, $x-\frac{1}{n}$ is an upper bound of S. But $x > x - \frac{1}{n}$; x can not then be the supremum of S, which is a contradiction. Thus $x^3 \not> 2$.

That leaves only one alternative (since we know $x \in \mathbb{R}$): $x^3 = 2$ or, equivalently, $x = u = \sqrt[3]{2} > 0$.

(We could also show it is irrational, but we'll leave it as an exercise.) ■

11. Let $S \subseteq \mathbb{R}$ and suppose that $s^* = \sup S$ belongs to S. If $u \notin S$, show that $\sup(S \cup \{u\}) = \sup\{s^*, u\}$.

Proof. In this case, we do not need to verify if s^* exists, as that is one of the hypotheses.

Set $v = \sup\{s^*, u\}$. Then, v is an upper bound of $S \cup \{u\}$ since $v \geq u$ and $v \geq s^* \geq s$ for all $s \in S$.

Furthermore, $v \in S \cup \{u\}$.

Hence, any upper bound of $S \cup \{u\}$ must be $\geq v$: consequently, v is the smallest upper bound of $\sup(S \cup \{u\})$. ∎

12. Show that a non-empty finite set $S \subseteq \mathbb{R}$ contains its supremum.

Solution. We use induction on the cardinality of S to show the result.

Base case: if $|S| = 1$, then $S = \{s_1\}$ for some $s_1 \in \mathbb{R}$. Clearly, $s_1 = \sup S \in S$.

Induction step: Suppose that the result holds for any set whose cardinality is $n = k$. Let S be any set with $|S| = k + 1$, say

$$S = \{s_1, \ldots, s_k, s_{k+1}\}.$$

Write $S = T \cup \{s_{k+1}\}$, with $T = \{s_1, \ldots, s_k\}$. Note that we can assume that $s_{k+1} \notin T$ (otherwise $|S| = k$).

Then T is non-empty and bounded since it is finite (exercise: a finite set is bounded); by completeness, $t^* = \sup T$ exists.

However, $|T| = k$. By the induction hypothesis, then, $\sup T \in T$, i.e. $t^* = s_j$ for some $j \in \{1, \ldots, k\}$.

According to the preceding problem,

$$\sup S = \sup(T \cup \{s_{k+1}\}) = \sup\{t^*, s_{k+1}\} \in T \cup \{s_{k+1}\} = S.$$

By induction, any non-empty finite set contains its supremum (and infimum too – it's the same idea). ■