MAT 2125
Elementary Real Analysis

Exercises – Solutions – Q16-Q20

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10. If $S \subseteq \mathbb{R}$ is a non-empty bounded set and $I_S = [\inf S, \sup S]$, show that $S \subseteq I_S$. Moreover, if J is any closed bounded interval of $\mathbb{R}$ such that $S \subseteq J$, show that $I_S \subseteq J$.

Proof. As S is non-empty and bounded, $\sup S$ and $\inf S$ exist by the completeness of $\mathbb{R}$.

Since $\inf S \leq s \leq \sup S$ for all $s \in S$, then $\inf S \leq \sup S$ and so the interval $I_S = [\inf S, \sup S]$ is well-defined.

Furthermore, the string of inequalities above also shows that $S \subseteq I_S$.

Now, let $J = [a, b]$ be a closed interval containing S. Then $a \leq s \leq b$ for all $s \in S$. Thus, a is a lower bound and b is an upper bound of S.

By definition,

$$a \leq \inf S \leq \sup S \leq b,$$

and so $I_S = [\inf S, \sup S] \subseteq [a, b] = J$. ∎

11. Prove that if $K_n = (n, \infty)$ for $n \in \mathbb{N}$, then

$$\bigcap_{n\in\mathbb{N}} K_n = \varnothing.$$

Solution. Suppose $x \in \bigcap K_n$. Then $x \in K_n$ for all n, i.e. $x > n$ for all $n \in \mathbb{N}$. This implies x is an upper bound of $\mathbb{N}$, which contradicts the Archimedean property. Hence, $\bigcap K_n = \varnothing$.

If you do not like contradiction proofs, here is the same proof, but presented as a direct proof.

Let $x \in \mathbb{R}$. We will show that $x \notin \bigcap K_n$; as x is arbitrary, this implies $\bigcap K_n = \varnothing$.

By the Archimedean property, there is a positive integer N such that $N > x$. Hence $x \notin K_n$ for all $n \geq N$. The conclusion follows. ■

12. If S is finite and $s^* \notin S$, show $S \cup \{s^*\}$ is finite.

Proof. If $S = \varnothing$, then $S \cup \{s^*\} = \{s^*\}$ is finite as the function $f : \mathbb{N}_1 \to \{s^*\}$ defined by $f(1) = s^*$ is a bijection.

Now, suppose $S \neq \varnothing$. As S is finite, there exist an integer k and a bijection $f : \mathbb{N}_k \to S$.

Define the associated function $\tilde{f} : \mathbb{N}_{k+1} \to S \cup \{s^*\}$ by

$$\tilde{f}(i) = \begin{cases} f(i) & \text{if } 1 \leq i \leq k \\ s^* & \text{if } i = k+1 \end{cases}.$$

As $s^* \notin S$, $\tilde{f}$ is a bijection. Hence $S \cup \{s^*\}$ is finite. ■

13. The first few terms of a sequence (x_n) are given below. Assuming that the "natural pattern" indicated by these terms persists, give a formula for the nth term x_n.

(a) $(5, 7, 9, 11, \ldots)$;

(b) $\left(\frac{1}{2}, -\frac{1}{4}, \frac{1}{8}, -\frac{1}{16}, \ldots\right)$;

(c) $\left(\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots\right)$;

(d) $(1, 4, 9, 16, \ldots)$.

Solution. There is no general method. This question is a wee bit on the easy side...

(a) Odd integers ≥ 5: $x_n = 2n + 3$ for all $n \geq 1$;

(b) Alternating powers of $\frac{1}{2}$: $x_n = (-1)^{n+1}\frac{1}{2^n}$ for all $n \geq 1$;

(c) Fractions where the denominator is one more than the numerator: $x_n = \frac{n}{n+1}$ for all $n \geq 1$;

(d) Perfect squares ≥ 1: $x_n = n^2$ for all $n \geq 1$. ■

14. Use the definition of the limit of a sequence to establish the following limits.

(a) $\lim_{n\to\infty}\left(\frac{1}{n^2+1}\right)=0;$

(b) $\lim_{n\to\infty}\left(\frac{2n}{n+1}\right)=2;$

(c) $\lim_{n\to\infty}\left(\frac{3n+1}{2n+5}\right)=\frac{3}{2}$, and

(d) $\lim_{n\to\infty}\left(\frac{n^2-1}{2n^2+3}\right)=\frac{1}{2}.$

Proof.

(a) Let $\varepsilon > 0$. By the Archimedean property, there is a positive integer $N_\varepsilon > \frac{1}{\varepsilon}$. Then

$$\left|\frac{1}{n^2+1} - 0\right| = \frac{1}{n^2+1} < \frac{1}{n^2} \leq \frac{1}{n} < \frac{1}{N_\varepsilon} < \varepsilon,$$

whenever $n > N_\varepsilon$.

(b) Let $\varepsilon > 0$. By the Archimedean property, there is a positive integer $N_\varepsilon > \frac{2}{\varepsilon}$. Then

$$\left|\frac{2n}{n+1} - 2\right| = \left|-\frac{2}{n+1}\right| = \frac{2}{n+1} < \frac{2}{n} < \frac{2}{N_\varepsilon} < \varepsilon,$$

whenever $n > N_\varepsilon$.

(c) Let $\varepsilon > 0$. By the Archimedean property, there is a positive integer $N_\varepsilon > \frac{13}{4} \cdot \frac{1}{\varepsilon}$. Then

$$\left|\frac{3n+1}{2n+5} - \frac{3}{2}\right| = \left|-\frac{13}{2(2n+5)}\right| = \frac{13}{2} \cdot \frac{1}{2n+5} < \frac{13}{2} \cdot \frac{1}{2n} = \frac{13}{4} \cdot \frac{1}{n} < \frac{13}{4} \cdot \frac{1}{N_\varepsilon},$$

which is smaller than ε whenever $n > N_\varepsilon$.

(d) Let $\varepsilon > 0$. By the Archimedean property, there is a positive integer $N_\varepsilon > \frac{5}{4} \cdot \frac{1}{\varepsilon}$. Then

$$\left|\frac{n^2-1}{2n^2+3} - \frac{1}{2}\right| = \left|-\frac{5}{2(2n^2+3)}\right| = \frac{5}{2} \cdot \frac{1}{2n^2+3} < \frac{5}{2} \cdot \frac{1}{2n^2} \leq \frac{5}{4} \cdot \frac{1}{n} < \frac{5}{4} \cdot \frac{1}{N_\varepsilon},$$

which is smaller than ε whenever $n > N_\varepsilon$. ■