MAT 2125
Elementary Real Analysis

Notes

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Theorem 1. (ARCHIMEDEAN PROPERTY) *Let* $x \in \mathbb{R}$. *Then* $\exists n_x \in \mathbb{N}^\times$ *such that* $x < n_x$.

Proof. Suppose that there is no such integer. Then $x \geq n \ \forall n \in \mathbb{N}$.

Consequently, x is an upper bound of $\mathbb{N}^\times$. But $\mathbb{N}^\times$ is a non-empty subset of $\mathbb{R}$. Since $\mathbb{R}$ is complete, $\alpha = \sup \mathbb{N}^\times$ exists.

By definition of the supremum (the smallest upper bound), $\alpha - 1$ is not an upper bound of $\mathbb{N}^\times$ (otherwise α would not be the smallest upper bound, as $\alpha - 1 < \alpha$ would be a smaller upper bound).

Since $\alpha - 1$ is not an upper bound of $\mathbb{N}^\times$, $\exists m \in \mathbb{N}^\times$ such that $\alpha - 1 < m$. Using the properties of $\mathbb{R}$, we must then have $\alpha < m + 1 \in \mathbb{N}^\times$; that is, α is not an upper bound of $\mathbb{N}^\times$.

This contradicts the fact that $\alpha = \sup \mathbb{N}^\times$, and so, since $\mathbb{N}^\times \neq \varnothing$, x cannot be an upper bound of $\mathbb{N}^\times$. Thus $\exists n_x \in \mathbb{N}^\times$ such that $x < n_x$. ■

Theorem 2. (VARIANTS; ARCHIMEDEAN PROPERTY) *Let* $x, y \in \mathbb{R}^+$*. Then* $\exists n_1, n_2, n_3 \geq 1$ *such that*

1. $x < n_1 y$;
2. $0 < \frac{1}{n_2} < y$, *and*
3. $n_3 - 1 \leq x < n_3$.

Proof.

1. Let $z = \frac{x}{y} > 0$. By the Archimedean property, $\exists n_1 \geq 1$ such that $z = \frac{x}{y} < n_1$. Then $x < n_1 y$.

2. If $x = 1$, then part 1 implies $\exists n_2 \geq 1$ such that $0 < 1 < n_2 y$. Then $0 < \frac{1}{n_2} < y$.

3. Let $L = \{m \in \mathbb{N}^\times : x < m\}$. By the Archimedean property, $L \neq \varnothing$. Indeed, there is at least one $n \geq 1$ such that $x < n$. By the well-ordering principle, L has a smallest element, say $m = n_3$. Then $n_3 - 1 \notin L$ (otherwise, $n_3 - 1$ would be the least element of L, which it is not) and so $n_3 - 1 \leq x < n_3$.

There are other variants, but these are the ones we'll use the most. ■

Theorem 4. (Cauchy's Inequality) *If $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ are real numbers, then*

$$\left(\sum a_i b_i\right)^2 \leq \left(\sum a_i^2\right)\left(\sum b_i^2\right).$$

(The indices are understood to run from 1 to n in what follows.) Furthermore, if $b_j \neq 0$ for one of $1 \leq j \leq n$, then equality holds if and only if $\exists s \in \mathbb{R}$ such that $a_i = sb_i$ for all $i = 1, \ldots, n$.

Proof. For any $t \in \mathbb{R}$,

$$0 \leq \sum (a_i + tb_i)^2 = \sum a_i^2 + 2t \sum a_i b_i + t^2 \sum b_i^2.$$

The right-hand side of this inequality is a polynomial of degree 2 in t.

It is always greater than or equal to 0: it has at most 1 real root, i.e. its discriminant

$$\left(2\sum a_ib_i\right)^2-4\left(\sum a_i^2\right)\left(\sum b_i^2\right)\leq 0,$$

and so

$$\left(\sum a_ib_i\right)^2\leq\left(\sum a_i^2\right)\left(\sum b_i^2\right).$$

If all the b_i are 0, the equality holds trivially, as both the left and right side of the Cauchy inequality are 0.

So suppose $b_i\neq 0$ for at least one of the values j between 1 and n. We have two statements to prove.

If $a_i = sb_i$ for all $i = 1, \ldots, n$ and $s \in \mathbb{R}$ is fixed then

$$\begin{aligned}\left(\sum a_i b_i\right)^2 &= \left(\sum s b_i^2\right)^2 = s^2 \left(\sum b_i^2\right)^2 = s^2 \left(\sum b_i^2\right)\left(\sum b_i^2\right) \\ &= \left(\sum s^2 b_i^2\right)\left(\sum b_i^2\right) = \left(\sum a_i^2\right)\left(\sum b_i^2\right).\end{aligned}$$

On the other hand, if

$$\left(\sum a_i b_i\right)^2 = \left(\sum a_i^2\right)\left(\sum b_i^2\right)$$

then

$$4\left(\sum a_i b_i\right)^2 - 4\left(\sum a_i^2\right)\left(\sum b_i^2\right) = 0.$$

But the left-hand side of this expression is the discriminant of the following polynomial of degree 2 in t:

$$\sum (a_i + tb_i)^2 = \sum a_i^2 + 2t \sum a_i b_i + t^2 \sum b_i^2.$$

Since the discriminant is 0, the polynomial has a unique root, say $t = -s$,

$$\therefore \sum (a_i - sb_i)^2 = 0.$$

Since $(a_i - sb_i)^2 \geq 0$ for all $i = 1, \ldots, n$, then

$$\begin{aligned}
(a_i - sb_i)^2 = 0 &\quad \text{for all } i = 1, \ldots, n \\
\therefore a_i - sb_i = 0 &\quad \text{for all } i = 1, \ldots, n \\
\therefore a_i = sb_i &\quad \text{for all } i = 1, \ldots, n. \qquad \blacksquare
\end{aligned}$$

Theorem 5. (Triangle Inequality) *If* $a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{R}$, *then*

$$\left(\sum (a_i + b_i)^2\right)^{1/2} \leq \left(\sum a_i^2\right)^{1/2} + \left(\sum b_i^2\right)^{1/2}.$$

Furthermore, if $b_j \neq 0$ *for one of* $1 \leq j \leq n$, *then equality holds if and only if* $\exists s \in \mathbb{R}$ *such that* $a_i = sb_i$ *for all* $i = 1, \ldots, n$.

Proof. As

$$\begin{aligned}
\sum (a_i + b_i)^2 &= \sum a_i^2 + 2\sum a_i b_i + \sum b_i^2 \\
\boxed{\text{Cauchy Ineq.}} &\leq \sum a_i^2 + 2\left(\sum a_i^2\right)^{1/2}\left(\sum b_i^2\right)^{1/2} + \sum b_i^2 \\
&= \left(\left(\sum a_i^2\right)^{1/2} + \left(\sum b_i^2\right)^{1/2}\right)^2.
\end{aligned}$$

Taking the square root on both sides yields the desired result.

If all the b_i are 0, the equality holds trivially, as both the left and right side of the Triangle Inequality are $\left(\sum a_i^2\right)^{1/2}$.

So suppose $b_i \neq 0$ for at least one of the values j between 1 and n. We have two statements to prove.

If $a_i = sb_i$ for all $i = 1, \ldots, n$ and $s \in \mathbb{R}$ is fixed then

$$\begin{aligned}\left(\sum (a_i + b_i)^2\right)^{1/2} &= \left(\sum (sb_i + b_i)^2\right)^{1/2} = \left(\sum (s+1)^2 b_i^2\right)^{1/2} \\ &= \left((s+1)^2 \sum b_i^2\right)^{1/2} = (s+1)\left(\sum b_i^2\right)^{1/2}\end{aligned}$$

and

$$\left(\sum a_i^2\right)^{1/2} + \left(\sum b_i^2\right)^{1/2} = \left(\sum s^2 b_i^2\right)^{1/2} + \left(\sum b_i^2\right)^{1/2}$$
$$= s\left(\sum b_i^2\right)^{1/2} + \left(\sum b_i^2\right)^{1/2} = (s+1)\left(\sum b_i^2\right)^{1/2}$$

and so equality holds.

On the other hand, if

$$\left(\sum (a_i+b_i)^2\right)^{1/2} = \left(\sum a_i^2\right)^{1/2} + \left(\sum b_i^2\right)^{1/2}$$

then

$$\sum (a_i+b_i)^2 = \left(\left(\sum a_i^2\right)^{1/2} + \left(\sum b_i^2\right)^{1/2}\right)^2.$$

Developing both sides of this expression yields

$$\sum a_i^2 + 2\sum a_ib_i + \sum b_i^2 = \sum a_i^2 + 2\left(\sum a_i^2\right)^{1/2}\left(\sum b_i^2\right)^{1/2} + \sum b_i^2,$$

or simply

$$\sum a_ib_i = \left(\sum a_i^2\right)^{1/2}\left(\sum b_i^2\right)^{1/2}.$$

Elevating both sides to the second power yields

$$\left(\sum a_ib_i\right)^2 = \left(\sum a_i^2\right)\left(\sum b_i^2\right).$$

By Theorem 4, $\exists s \in \mathbb{R}$ such that $a_i = sb_i$ for all $i = 1, \ldots, n$. ■

Theorem 7. (Density of $\mathbb{Q}$) *Let $x, y \in \mathbb{R}$ such that $x < y$. Then, $\exists r \in \mathbb{Q}$ such that $x < r < y$.*

Proof. There are three distinct cases.

1. If $x < 0 < y$, then select $r = 0$.

2. If $0 \leq x < y$, then $y - x > 0$ and $\frac{1}{y-x} > 0$.

   By the Archimedean property, $\exists n \geq 1$ such that

   $$n > \frac{1}{y - x} > 0.$$

   By that same property, $\exists m \geq 1$ such that $m - 1 \leq nx < m$. Since $n(y - x) > 1$, then $ny - 1 > nx$ and $nx \geq m - 1$.

By transitivity of $<$, $ny-1>m-1$, that is $ny>m$. But $m>nx$, so $ny>m>nx$ and $y>\frac{m}{n}>x$. Select $r=\frac{m}{n}$.

3. If $x<y\leq 0$, then $y-x>0$ and $\frac{1}{y-x}>0$. By the Archimedean property, $\exists n\geq 1$ such that
$$n>\frac{1}{y-x}>0.$$
Note that $-nx>0$. By yet another variant of that property (that we haven't explicitly stated in class, but it's not too much work to show it), $\exists m\geq 0$ such that $m<-nx\leq m+1$ or $-m-1\leq nx<-m$.

Since $n(y-x)>1$, then $ny-1>nx$ and $nx\geq -m-1$.

By transitivity of $<$, $ny-1>-m-1$, that is $ny>-m$. But $-m>nx$, so $ny>-m>nx$ and $y>-\frac{m}{n}>x$. Select $r=-\frac{m}{n}$. ■

Theorem 14. (Operations on Sequences and Limits) *Let* $(x_n), (y_n)$ *be convergent sequences, with* $x_n \to x$ *and* $y_n \to y$. *Let* $c \in \mathbb{R}$. *Then*

1. $|x_n| \to |x|$;
2. $(x_n + y_n) \to (x + y)$;
3. $x_n y_n \to xy$ *and* $cx_n \to cx$;
4. $\frac{x_n}{y_n} \to \frac{x}{y}$, *if* $y_n, y \neq 0$ *for all* n.

Proof. We show each part using the definition of the limit of a sequence.

1. Let $\varepsilon > 0$. As $x_n \to x$, $\exists N'_\varepsilon$ such that $|x_n - x| < \varepsilon$ whenever $n > N'_\varepsilon$. But $||x_n| - |x|| \leq |x_n - x|$, according to theorem 6. Hence, for $\varepsilon > 0$, $\exists N_\varepsilon = N'_\varepsilon$ such that

$$||x_n| - |x|| \leq |x_n - x| < \varepsilon$$

   whenever $n > N_\varepsilon$, i.e. $|x_n| \to |x|$.

2. Let $\varepsilon > 0$. Then $\frac{\varepsilon}{2} > 0$. As $x_n \to x$ and $y_n \to y$, $\exists N^x_{\frac{\varepsilon}{2}}, N^y_{\frac{\varepsilon}{2}}$ such that

$$|x_n - x| < \frac{\varepsilon}{2} \quad \text{and} \quad |y_n - y| < \frac{\varepsilon}{2} \tag{1}$$

   whenever $n > N^x_{\frac{\varepsilon}{2}}$ and $n > N^y_{\frac{\varepsilon}{2}}$ respectively. Set $N_\varepsilon = \max\left\{N^x_{\frac{\varepsilon}{2}}, N^y_{\frac{\varepsilon}{2}}\right\}$.

Then, whenever $n > N_\varepsilon$ (so whenever n is strictly larger than $N^x_{\varepsilon/2}$ and $N^y_{\varepsilon/2}$ *at the same time*),

$$\begin{aligned} |(x_n + y_n) - (x + y)| = |(x_n - x) + (y_n - y)| &\leq |x_n - x| + |y_n - y| \\ \boxed{\text{by (1)}} &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \end{aligned}$$

i.e. $(x_n + y_n) \to (x + y)$.

3. According to theorem 13, (x_n) and (y_n) are bounded since they are convergent sequences. Then $\exists M_x, M_y \in \mathbb{N}$ such that

$$|x_n| < M_x \quad \text{and} \quad |y_n| < M_y$$

for all n.

Let $\varepsilon > 0$. Then $\frac{\varepsilon}{2M_x}, \frac{\varepsilon}{2M_y} > 0$. As $x_n \to x$ and $y_n \to y$, $\exists N^x_{\frac{\varepsilon}{2M_y}}, N^y_{\frac{\varepsilon}{2M_x}} \in \mathbb{N}$ such that

$$|x_n - x| < \frac{\varepsilon}{2M_y} \quad \text{and} \quad |y_n - y| < \frac{\varepsilon}{2M_x} \tag{2}$$

whenever $n > N^x_{\frac{\varepsilon}{2M_y}}$ and $n > N^y_{\frac{\varepsilon}{2M_x}}$ respectively. Moreover, $|y| \leq M_y$ (otherwise $\frac{|y|-M_y}{2} > 0$. Then, for $\varepsilon = \frac{|y|-M_y}{2}$, we get

$$|y_n - y| \geq ||y| - |y_n|| \geq |y| - M_y = 2\varepsilon > \varepsilon$$

for all $n \in \mathbb{N}$, which contradicts the definition of $y_n \to y$).

Set $N_\varepsilon = \max\left\{N^x_{\frac{\varepsilon}{2M_x}}, N^y_{\frac{\varepsilon}{2M_y}}\right\}$. Then, whenever $n > N_\varepsilon$,

$$\begin{aligned}
|x_ny_n - xy| = |x_ny_n - x_ny + x_ny - xy| &= |x_n(y_n - y) + y(x_n - x)| \\
&\leq |x_n||y_n - y| + |y||x_n - x| \\
&< M_x|y_n - y| + M_y|x_n - x| \\
\boxed{\text{by (2)}} \quad &< M_x\frac{\varepsilon}{2M_x} + M_y\frac{\varepsilon}{2M_y} \\
&= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,
\end{aligned}$$

i.e. $x_ny_n \to xy$.

Furthermore, if the sequence (y_n) is given by $y_n = c$ for all n, then the preceding result yields $cx_n \to cx$, since $y_n = c \to c$ (You should show this).

4. It is enough to show $\frac{1}{y_n} \to \frac{1}{y}$ under the hypotheses above; then the result will hold by part 3.

   Since $y \neq 0$, $\frac{|y|}{2} > 0$. Hence, as $y_n \to y$, $\exists N_{|y|/2} \in \mathbb{N}$ such that $|y_n - y| < \frac{|y|}{2}$, whenever $n > N_{\frac{|y|}{2}}$. According to theorem 6,

$$|y| - |y_n| < |y - y_n| < \frac{|y|}{2}, \quad \text{and so}$$

$$\frac{|y|}{2} < |y_n| \quad \text{or} \quad \frac{1}{|y_n|} < \frac{2}{|y|} \tag{3}$$

   whenever $n > N_{|y|/2}$ (these expressions make sense as neither y_n nor y is 0 for all n).

Let $\varepsilon > 0$. Then $|y|^2\frac{\varepsilon}{2} > 0$. As $y_n \to y$, $\exists N_{|y|^2\frac{\varepsilon}{2}} \in \mathbb{N}$ such that

$$|y_n - y| < |y|^2\frac{\varepsilon}{2} \tag{4}$$

whenever $n > N_{|y|^2\frac{\varepsilon}{2}}$. Set $N_\varepsilon = \max\left\{N_{\frac{|y|}{2}}, N_{|y|^2\frac{\varepsilon}{2}}\right\}$. Then, whenever $n > N_\varepsilon$,

$$\begin{aligned} \left|\frac{1}{y_n} - \frac{1}{y}\right| = \left|\frac{y - y_n}{y_n y}\right| &= \frac{|y - y_n|}{|y_n y|} \\ \boxed{\text{by (3)}} &< \frac{2|y - y_n|}{|y|^2} \\ \boxed{\text{by (4)}} &< \frac{2}{|y|^2} \cdot |y|^2\frac{\varepsilon}{2} = \varepsilon, \quad \text{i.e. } \frac{1}{y_n} \to= \frac{1}{y}. \qquad \blacksquare \end{aligned}$$

Theorem 56. (Composition Theorem for Integrals)
Let $I = [a, b]$ and $J = [\alpha, \beta]$, $f : I \to \mathbb{R}$ Riemann-integrable on I, $\varphi : J \to \mathbb{R}$ continuous on J and $f(I) \subseteq J$. Then $\varphi \circ f : I \to \mathbb{R}$ is Riemann-integrable on I.

Proof. Let $\varepsilon > 0$, $K = \sup\{|\varphi(x)| \mid x \in J\}$ (guaranteed to exist by the Max/Min theorem) and $\varepsilon' = \frac{\varepsilon}{b-a+2K}$.

Since φ is uniformly continuous on J (being continuous on a closed, bounded interval), $\exists \delta_\varepsilon > 0$ s.t.

$$|x - y|\delta_\varepsilon,\ x, y, \in J \implies |\varphi(x) - \varphi(y)| < \varepsilon'.$$

Without loss of generality, pick $\delta_\varepsilon < \varepsilon'$.

Since f is Riemann-integrable on I, $\exists P = \{x_0, \ldots, x_n\}$ a partition of $I = [a, b]$ s.t.

$$U(P; f) - L(P; f) < \delta_\varepsilon^2$$

(according to Riemann's criterion).

We show that $U(P; \varphi \circ f) - L(P; \varphi \circ f) < \varepsilon$, and so that $\varphi \circ f$ is Riemann-integrable according ti Riemann's criterion.

Over $[x_{i-1}, x_i]$ for $i = 1, \ldots, n$, set

$$m_i = \inf\{f(x)\},\ M_i = \sup\{f(x)\},\ \tilde{m}_i = \inf\{\varphi(f(x))\},\ \tilde{M}_i = \sup\{\varphi(f(x))\}.$$

With those, set $A = \{i \mid M_i - m_i < \delta_\varepsilon\}$, $B = \{i \mid M_i - m_i \geq \delta_\varepsilon\}$.

- If $i \in A$, then

$$x, y \in [x_{i-1}, x_i] \implies |f(x) - f(y)| \leq M_i - m_i < \delta_\varepsilon,$$

so $|\varphi(f(x)) - \varphi(f(y)| < \varepsilon' \ \forall x, y \in [x_{i-1}, x_i]$. In particular, $\tilde{M}_i - \tilde{m}_i \leq \varepsilon'$.

- If $i \in B$, then

$$x, y \in [x_{i-1}, x_i] \implies |\varphi(f(x)) - \varphi(f(y))| \leq |\varphi(f(x))| + |\varphi(f(y))| \leq 2K.$$

In particular, $\tilde{M}_i - \tilde{m}_i \leq 2K$, since $-K \leq \tilde{m}_i \leq \varphi(z) \leq \tilde{M}_i \leq K$ for all $z \in [x_{i-1}, x_i]$.

Then

$$
\begin{aligned}
U(P;\varphi\circ f)-L(P;\varphi\circ f)&=\sum_{i=1}^{n}(\tilde{M}_i-\tilde{m}_i)(x_i-x_{i-1})\\
&=\sum_{i\in A}(\tilde{M}_i-\tilde{m}_i)(x_i-x_{i-1})+\sum_{i\in B}(\tilde{M}_i-\tilde{m}_i)(x_i-x_{i-1})\\
&\leq \varepsilon'\sum_{i\in A}(x_i-x_{i-1})+2K\sum_{i\in B}(x_i-x_{i-1})\\
&\leq \varepsilon'(b-a)+2K\sum_{i\in B}\frac{(M_i-m_i)}{\delta_\varepsilon}(x_i-x_{i-1})\\
&\varepsilon'(b-a)+\frac{2K}{\delta_\varepsilon}\sum_{i=1}^{n}(M_i-m_i)(x_i-x_{i-1}).
\end{aligned}
$$

By earlier work in the proof, we have

$$\sum_{i=1}^{n}(M_i - m_i)(x_i - x_{i-1}) \leq U(P;f) - L(P;f) < \delta_\varepsilon^2,$$

so that

$$\begin{aligned} U(P;\varphi\circ f) - L(P;\varphi\circ f) &< \varepsilon'(b-a) + \frac{2K}{\delta_\varepsilon}\cdot\delta_\varepsilon^2 \\ &= \varepsilon'(b-a) + 2K\delta_\varepsilon < \varepsilon'(b-a) + 2K\varepsilon' \\ &= \varepsilon'(b-a+2K) = \varepsilon, \end{aligned}$$

which completes the proof. ■