MAT 2125 - Some Numbered Results (and Corollaries)

Theorem 1 (Archimedean Property)
Let $x \in \mathbb{R}$. Then $\exists n_x \in \mathbb{N}^\times$ such that $x < n_x$.

Theorem 2 (Archimedean Property; Variants)
Let $x, y \in \mathbb{R}^+$. Then $\exists n_1, n_2, n_3 \in \mathbb{N}^\times$ such that

1. $x < n_1 y$;
2. $0 < \frac{1}{n_2} < y$, *and*
3. $n_3 - 1 \leq x < n_3$.

Theorem 3 (Bernoulli's Inequality)
Let $x \geq -1$. Then $(1+x)^n \geq 1 + nx$, $\forall n \in \mathbb{N}$.

Theorem 4 (Cauchy's Inequality)
If $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ are real numbers, then

$$\left(\sum a_i b_i\right)^2 \leq \left(\sum a_i^2\right)\left(\sum b_i^2\right).$$

(The indices are understood to run from 1 to n in what follows.) Furthermore, if $b_j \neq 0$ for one of $1 \leq j \leq n$, then equality holds if and only if $\exists s \in \mathbb{R}$ such that $a_i = s b_i$ for all $i = 1, \ldots, n$.

Theorem 5 (Triangle Inequality)
If $a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{R}$, then

$$\left(\sum (a_i + b_i)^2\right)^{1/2} \leq \left(\sum a_i^2\right)^{1/2} + \left(\sum b_i^2\right)^{1/2}.$$

Furthermore, if $b_j \neq 0$ for one of $1 \leq j \leq n$, then equality holds if and only if $\exists s \in \mathbb{R}$ such that $a_i = s b_i$ for all $i = 1, \ldots, n$.

Theorem 6 (Properties of the Absolute Value)
If $x, y \in \mathbb{R}$ and $\varepsilon > 0$, then

1. $|x| = \sqrt{x^2}$
2. $-|x| \leq x \leq |x|$
3. $|xy| = |x||y|$
4. $|x + y| \leq |x| + |y|$
5. $|x - y| \leq |x| + |y|$
6. $||x| - |y|| \leq |x - y|$
7. $|x - y| < \varepsilon \iff y - \varepsilon < x < y + \varepsilon$

Theorem 7 (Density of $\mathbb{Q}$, $\mathbb{R} \setminus \mathbb{Q}$)
Let $x, y \in \mathbb{R}$ such that $x < y$. Then, $\exists r \in \mathbb{Q}, z \notin \mathbb{Q}$ such that $x < r < y$ and $x < z < y$.

Theorem 12 (Unique Limit)
A convergent sequence of real numbers $(x_n)_{n\in\mathbb{N}}$ has exactly one limit.

Theorem 13 *Any convergent sequence of real numbers $(x_n)_{n\in\mathbb{N}}$ is bounded.*

Theorem 14 (Operations on Sequences and Limits)
Let $(x_n), (y_n)$ be convergent sequences, with $x_n \to x$ and $y_n \to y$. Let $c \in \mathbb{R}$. Then

1. $|x_n| \to |x|$;
2. $(x_n + y_n) \to (x + y)$;
3. $x_n y_n \to xy$ and $cx_n \to cx$;
4. $\frac{x_n}{y_n} \to \frac{x}{y}$, if $y_n, y \neq 0$ for all n.

Theorem 15 (Comparison Theorem for Sequences)
Let $(x_n), (y_n)$ be convergent sequences of real numbers with $x_n \to x$, $y_n \to y$, and $x_n \leq y_n$ $\forall n \in \mathbb{N}$. Then $x \leq y$.

Theorem 16 (Squeeze Theorem for Sequences)
Let $(x_n), (y_n), (z_n) \subseteq \mathbb{R}$ be such that $x_n, z_n \to \alpha$ and $x_n \leq y_n \leq z_n$, $\forall n \in \mathbb{N}$. Then $y_n \to \alpha$.

Theorem 17 *Let $x_n \to x$. If $x_n \geq 0$ $\forall n \in \mathbb{N}$, then $\sqrt{x_n} \to \sqrt{x}$.*

Theorem 32 *If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$, then f is bounded on $[a, b]$.*

Theorem 33 (Max/Min Theorem)
If $f : [a, b] \to \mathbb{R}$ is continuous, then f reaches a global maximum and a global minimum of $[a, b]$.

Theorem 34 *Let $f : [a, b] \to \mathbb{R}$ be continuous. If $\exists \alpha, \beta \in [a, b]$ such that $f(\alpha)f(\beta) < 0$, then $\exists \gamma \in (a, b)$ such that $f(\gamma) = 0$.*

Theorem 35 (Intermediate Value Theorem)
Let $f : [a, b] \to \mathbb{R}$ be continuous. If $\exists \alpha < \beta \in [a, b]$ such that $f(\alpha) < k < f(\beta)$ or $f(\alpha) > k > f(\beta)$, then $\exists \gamma \in (a, b)$ such that $f(\gamma) = k$.

Theorem 36 *If $f : [a, b] \to \mathbb{R}$ is continuous, then $f([a, b])$ is a closed and bounded interval.*

Theorem 37 *If $f : A \to \mathbb{R}$ is uniformly continuous on A, then f is continuous on A.*

Theorem 38 *Let $f : [a, b] \to \mathbb{R}$. Then f is uniformly continuous on $[a, b]$ if and only if f is continuous on $[a, b]$.*

Theorem 51 *Let $I = [a, b]$ and f be bounded on I. Then the lower integral and upper integral of f on I satisfy $L(f) \leq U(f)$.*

Theorem 52 (Riemann's Criterion)
Let $I = [a, b]$ and $f : I \to \mathbb{R}$ be a bounded function. Then f is Riemann-integrable if and only if $\forall \varepsilon > 0$, $\exists P_\varepsilon$ a partition of I such that the lower sum and the upper sum of f corresponding to P_ε satisfy $U(P_\varepsilon; f) - L(P_\varepsilon; f) < \varepsilon$.

Theorem 53 *Let $I = [a, b]$ and $f : I \to \mathbb{R}$ be a monotone function on I. Then f is Riemann-integrable on I.*

Theorem 54 *Let $I = [a, b]$ and $f : I \to \mathbb{R}$ be continuous on I. Then f is Riemann-integrable on I.*

Theorem 55 (Properties of the Riemann Integral)
Let $I = [a, b]$ and $f, g : I \to \mathbb{R}$ be Riemann-integrable on I. Then

1. *$f + g$ is Riemann-integrable on I, with $\int_a^b (f + g) = \int_a^b f + \int_a^b g$;*
2. *if $k \in \mathbb{R}$, $k \cdot f$ is Riemann-integrable on I, with $\int_a^b k \cdot f = k \int_a^b f$;*
3. *if $f(x) \leq g(x)$ $\forall x \in I$, then $\int_a^b f \leq \int_a^b g$, and*
4. *if $|f(x)| \leq K$ $\forall x \in I$, then $\left|\int_a^b f\right| \leq K(b - a)$.*

Theorem 56 (Additivity the Riemann Integral)
Let $I = [a, b]$, $c \in (a, b)$, and $f : I \to \mathbb{R}$ be bounded on I. Then f is Riemann-integrable on I if and only if it is Riemann-integrable on $I_1 = [a, c]$ and on $I_2 = [c, b]$. When that is the case, $\int_a^b f = \int_a^c f + \int_c^b f$.

Theorem 57 (Composition Theorem for Integrals)
Let $I = [a, b]$ and $J = [\alpha, \beta]$, $f : I \to \mathbb{R}$ Riemann-integrable on I, $\varphi : J \to \mathbb{R}$ continuous on J and $f(I) \subseteq J$. Then $\varphi \circ f : I \to \mathbb{R}$ is Riemann-integrable on I.

Theorem 58 *Let $I = [a, b]$ and $f, g : I \to \mathbb{R}$ be Riemann-integrable on I. Then fg and $|f|$ are Riemann-integrable on I, and $\left|\int_a^b f\right| \leq \int_a^b |f|$.*

Theorem 73 (Cauchy's Criterion for Series of Functions)
Let $I = [a, b]$ and $f_n : I \to \mathbb{R}$, $\forall n \in \mathbb{N}$. Then $\displaystyle\sum_{n=1}^{\infty} f_n \rightrightarrows f$ on I if and only if $\forall \varepsilon > 0$, $\exists N_\varepsilon \in \mathbb{N}$ (independent of x) such that $\displaystyle\left|\sum_{i=n+1}^{m}\right| < \varepsilon$ whenever $m > n > N_\varepsilon$ $\forall x \in I$.

Theorem 74 (Weierstrass's $M-$Test)
Let $I = [a, b]$, $f_n : I \to \mathbb{R}$, and $M_n > 0$ $\forall n \in \mathbb{N}$. If $|f_n(x)| \leq M_n$ $\forall x \in I$, $\forall n \in \mathbb{N}$, and if $\sum_{n=1}^{\infty} M_n$ converges, then $\displaystyle\sum_{n=1}^{\infty} f_n$ converges uniformly on I.