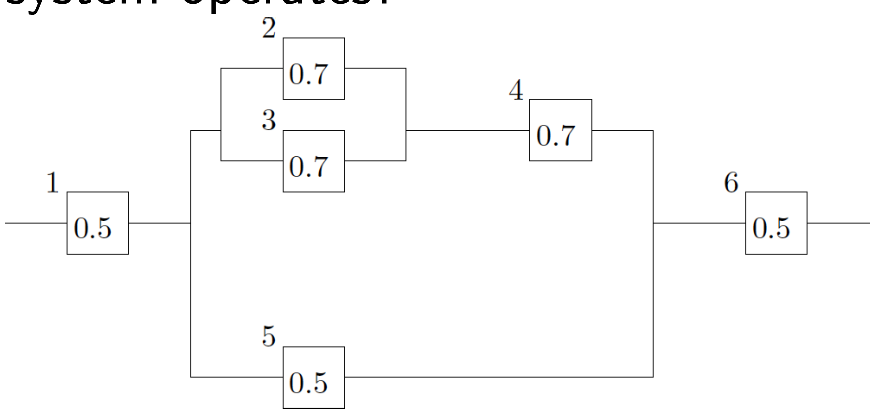
Q12. Consider the following system with six components. We say that it is functional if there exists a path of functional components from left to right. The probability of each component functions is shown. Assume that the components function or fail independently. What is the probability that the system operates?



a) 0.1815 b) 0.8185 c) 0.6370 d) 0.2046 e) none of the preceding

Solution: let Box A consist of components $2, 3, 4, 5$, Box B consist of components $2, 3, 4$, and Box C consist of components $2, 3$. We will denote the probability that Box j operates by $P(j)$, $j \in \{A, B, C\}$.

We are interested in the probability $P(S)$ that the system operates. Because the components function or fail independently,

$$\begin{aligned} P(S) &= P(\text{component } 1 \text{ and Box } A \text{ and component } 6 \text{ operate}) \\ &= P(1 \text{ operates}) \times P(A) \times P(6 \text{ operates}) \\ &= 0.5 \cdot P(A) \cdot 0.5 = 0.5^2 P(A). \end{aligned}$$

There are two ways for Box A to operate: either component 5 operates (with probability 0.5) or Box B operates:

$$
\begin{aligned}
P(A) &= P(\text{component } 5 \text{ operates or Box } B \text{ operates}) \\
&= P(5 \text{ operates}) + P(B) \\
&\qquad\qquad - P(\text{component } 5 \text{ operates and Box } B \text{ operates}) \\
&= P(5 \text{ operates}) + P(B) - P(5 \text{ operates})P(B) \\
&= 0.5 + P(B) - 0.5P(B) = 0.5(1 + P(B)).
\end{aligned}
$$

Thus,

$$P(S) = 0.5^2 P(A) = 0.5^2 \cdot 0.5(1 + P(B) = 0.5^3(1 + P(B)).$$

In order for Box B to operate, we need both Box C and component 4 to operate, so that

$$\begin{aligned} P(B) &= P(\text{Box } C \text{ operates and component } 4 \text{ operates}) \\ &= P(4 \text{ operates})P(C) = 0.7P(C). \end{aligned}$$

Thus,

$$P(S) = 0.5^3(1 + P(B)) = 0.5^3(1 + 0.7P(C)).$$

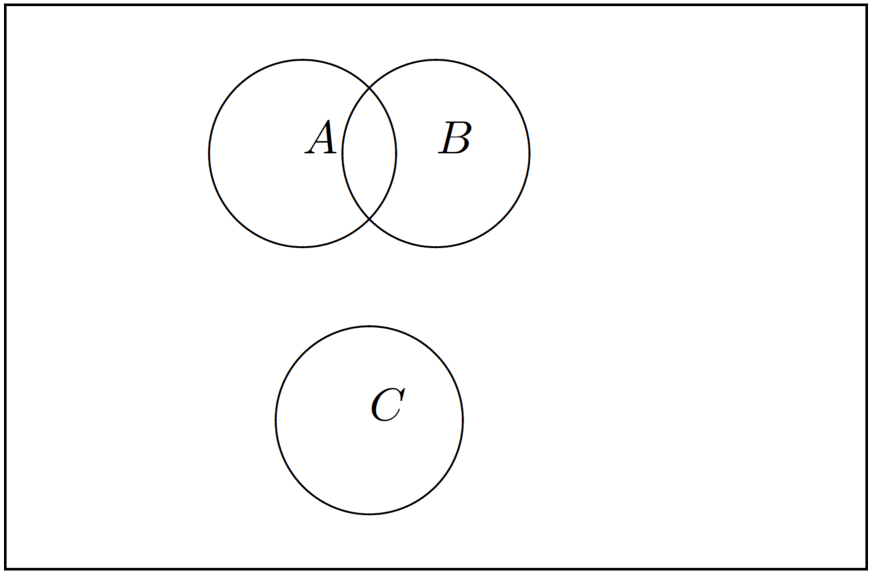
Finally, there are two ways for Box C to operate: either component 2 operates (with probability 0.7) or component 3 operates (also with probability 0.7):

$$\begin{aligned}
P(C) &= P(\text{component } 2 \text{ operates or component } 3 \text{ operates}) \\
&= P(2 \text{ operates}) + P(3 \text{ operates}) \\
&\qquad\qquad - P(\text{components } 2 \text{ operates and component } 3 \text{ operates}) \\
&= P(2 \text{ operates}) + P(3 \text{ operates}) - P(2 \text{ operates})P(3 \text{ operates}) \\
&= 0.7 + 0.7 - 0.7 \cdot 0.7 = 0.7(2 - 0.7) = 0.7 \cdot 1.3.
\end{aligned}$$

Thus,

$$\begin{aligned}
P(S) &= 0.5^3(1 + 0.7P(C)) = 0.5^3(1 + 0.7 \cdot 0.7 \cdot 1.3) \\
&= 0.5^3(1 + 0.7^2 \cdot 1.3) = 0.24046.
\end{aligned}$$

Q13. Three events are shown in the Venn diagram below.



Shade the region corresponding to the following events:

a) A^c

b) $(A \cap B) \cup (A \cap B^c)$

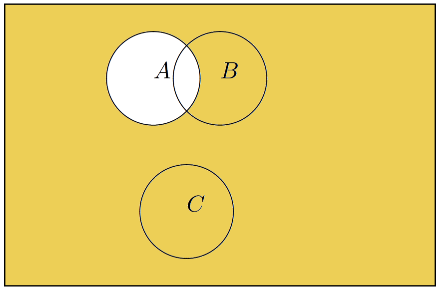
c) $(A \cap B) \cup C$

d) $(B \cup C)^c$

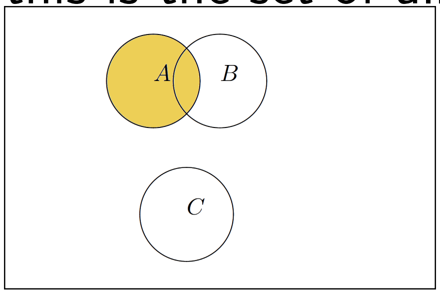
e) $(A \cap B)^c \cup C$

Solution:

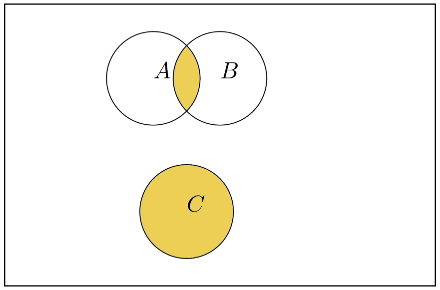
a) This is the set of all outcomes not in A:



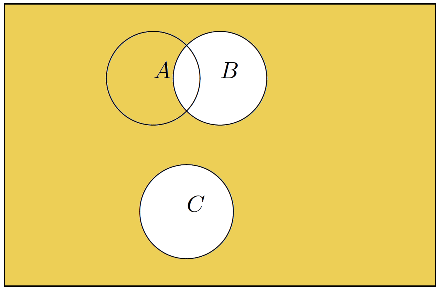
b) We can rewrite $(A \cap B) \cup (A \cap B^c) = A \cap (B \cup B^c) = A \cap \mathcal{S} = A$, so this is the set of all outcomes in A:



c) This is the set all outcomes either in C or in the intersection of A and B:



d) This is the set of all outcomes that are not in either B or C (or that are neither in B nor in C):



e) Since C and $A \cup B$ are mutually exclusive (disjoint), $C \subseteq (A \cap B)^c$ and $(A \cap B)^c \cup C = (A \cap B)^c$, so this is the set of all outcomes not in $A \cap B$: