uOttawa V1 Department of Mathematics and Statistics

MAT 2377
Winter 2020
Midterm Examination
February 11, 2020
Time Limit: 80 Minutes

Name: ______________

Student ID: ______________

SOLUTIONS

This exam contains 13 pages (including this cover page, the formula page, and the distribution tables) and 10 questions.

This is a closed book examination. Graphing and programmable calculators are not allowed; as are electronic devices of any kind (except for the Faculty of Science-approved calculators).

For questions 1 to 8, **only the final answer** of each question or sub-question will be graded; no part marks are available. Part marks may be achieved for questions 9 and 10, in which you need to justify fully your answers.

Grade Table (for professor use only)

Question:	1	2	3	4	5	6	7	8	9	10	Total
Points:	1	1	1	1	1	4	6	4	3	3	25
Score:	D	C	B	B	B						
other version:	B	A	F	F	E						

1. [1 point] Samples of 20 parts from a metal punching process are selected every hour. Typically, 1% of the parts require re-work. Let X denote the number of parts in the sample that require re-work. A process problem is suspected if X exceeds its mean by more than three standard deviations. To the nearest thousandth, what is the probability that there is a process problem?

A. 0.198 B. 0.445 C. 0.983 D. 0.017 E. 0.555 F. other

We have $X \sim B(n,p)$, with $n=20$, $p=0.01$.

By definition

$\mu_X = E[X] = np = 0.2$

$Var[X] = np(1-p) = 0.198$

$SD[X] = \sqrt{Var[X]} = 0.44$

We are looking for

$P(X > E[X] + 3SD[X])$

$= P(X > 0.2 + 3 \cdot 0.44) = P(X > 1.535)$

$= P(X \geq 2) = 1 - P(X \leq 1)$

$= 1 - P(X=0) - P(X=1)$

$= 1 - \binom{20}{0}(0.01)^0(0.99)^{20} - \binom{20}{1}(0.01)^1(0.99)^{19}$

$= 0.017$

Answer: D / B

v1 v2

2. [1 point] A medical research team wished to evaluate a proposed screening test for Alzheimer's disease. The test was given to a random sample of 450 patients with Alzheimer's disease; in 436 cases the test result was positive. The test was also given to a random sample of 500 patients without the disease; only in 5 cases was the result was positive. It is known that in Canada 11.3% of the population aged 65+ have Alzheimer's disease. Find the probability that a person has the disease given that their test was positive (choose the closest answer).

A. 0.97 B. 0.03 C. 0.93 D. 0.07 E. 0.99 F. 0.01 G. other

A: test is positive
D: person has disease

$P(A|D) = \frac{436}{450}$

$P(A|\bar{D}) = \frac{5}{500}$

$P(D) = 0.113$

Bayes $\Rightarrow$

$$P(D|A) = \frac{P(A|D)P(D)}{P(A|D)P(D) + P(A|\bar{D})P(\bar{D})}$$

$$= \frac{436/450\,(0.113)}{436/450\,(0.113) + 5/500\,(1-0.113)} = 0.925$$

Answer: $\boxed{C}$ / A

V1 V2

3. [1 point] Consider a random variable X with probability density function given by

$$f(x) = \begin{cases} 0 & \text{if } x \leq -1 \\ 0.75(1-x^2) & \text{if } -1 \leq x < 1 \\ 0 & \text{if } x \geq 1 \end{cases}$$

What is the expected value and the standard deviation of X?

A. 0, 3 B. 0, 0.447 C. 1, 0.2 D. 1, 3 E. 0, 0.2 F. 1, 0.447 G. other

$$E[X] = \int_{-\infty}^{\infty} x f(x)\,dx = 0.75\int_{-1}^{1} x(1-x^2)\,dx = 0.75\int_{-1}^{1}(x-x^3)\,dx = 0$$

$$Var[X] = E[X^2] - E^2[X] = E[X^2] - 0 = E[X^2] = \int_{-\infty}^{\infty} x^2 f(x)\,dx$$

$$= \int_{-1}^{1} 0.75\,x^2(1-x^2)\,dx = \int_{-1}^{1} 0.75\,(x^2-x^4)\,dx = 2(0.75)\left[\frac{2}{15}\right] = 0.2$$

$$SD[X] = \sqrt{Var[X]} = \sqrt{0.2} \approx 0.447$$

Answer: $\boxed{B}$ / F

V1 V2

4. [1 point] In the inspection of tin plate produced by a continuous electrolytic process, 0.2 imperfections are spotted per minute, on average. Find the probability of spotting at least 2 imperfections in 5 minutes. Assume that we can model the occurrences of imperfections as a Poisson process.

A. 0.736 B. 0.264 C. 0.632 D. 0.368 E. 0.483 F. 0.527 G. other

X: # of imperfections found in 5 min

Spotting rate $= \frac{0.2}{\text{min}} \times 5 \text{ min} = 1/5 \text{ min}$

$X \sim P(\lambda)$, $\lambda = 1$

So

$P(X \geq 2) = 1 - P(X \leq 1) = 1 - P(X=0) - P(X=1)$

$= 1 - e^{-\lambda} - \lambda e^{-\lambda} \underset{\lambda=1}{=} 1 - 2e^{-1} \approx 0.264$

Answer: B / F

V1 V2

5. [1 point] Let $X \sim \text{Exp}(\lambda)$ with mean 10. What is $P(X > 30 | X > 10)$ equal to?

A. $1 - \exp(-2)$ B. $\exp(-2)$ C. $\exp(-3)$ D. $1/10$ E. $\exp(-200)$ F. other

We have $10 = E[X] = \frac{1}{\lambda}$, so $\lambda = \frac{1}{10}$

Because the exponential is memory-less,

$P(X > 30 | X > 10) = P(X > 30-10) = P(X > 20)$

$= 1 - P(X < 20) = 1 - (1 - \exp(-\lambda \cdot 20))$

$= \exp(-2)$

Answer: B / E

V1 V2

6. A new robotic arm is tested prior to being marketed to the general public. The test results on 50 independent samples are shown below:

	Mechanical Defect	No Mechanical Defect
Electrical Defect	5	10
No Electrical Defect	5	30

Assume that these relative proportions are perfectly representative of the product (and of its flaws) outside the sample.

(a) [1 point] What is the probability that a robotic arm has a mechanical defect?

V1. $P(M) = \frac{5+5}{50} = \frac{10}{50} = \frac{1}{5} = 0.2$

V2. Same

(b) [1 point] What is the probability that a robotic arm has neither a mechanical nor an electrical defect?

V1. $P(M^c \cap E^c) = P((M \cup E)^c) = \frac{30}{50} = 0.6$

V2. $\frac{25}{50} = 0.5$

(c) [1 point] What is the probability that a robotic arm has a mechanical defect if it is known that it has an electrical defect?

V1. $P(M|E) = \frac{P(M \cap E)}{P(E)} = \frac{5}{15} \simeq 0.33$

V2. $P(M|E) = P(M \cap E)/P(E) = \frac{5}{20} = 0.25$

(d) [1 point] What is the probability that each of 3 robotic arms purchased by a company has at least one defect (assume that defects occur independently)?

V1. $P(M \cup E) = P(M) + P(E) - P(M \cap E) = \frac{10}{50} + \frac{15}{50} - \frac{5}{50} = 0.4$

Robot 1 0.4 Robot 2 0.4 Robot 3 0.4

total $= (0.4)^3 = 0.064$

V2. $P(M \cup E) = P(M) + P(E) - P(M \cap E) = \frac{10}{50} + \frac{20}{50} - \frac{5}{50} = 0.5$

Robot 1 0.5 Robot 2 0.5 Robot 3 0.5

Total $= (0.5)^3 = 0.125$

7. Let X be a random variable with probability mass function

$$P(X = x) = \frac{2x+1}{25}, \quad x = 0, 1, 2, 3, 4.$$

(a) [3 points] Evaluate: $P(1 \le X < 4)$, $P(X > -2)$ et $P(X \le 1)$.

$P(1 \le X < 4) = P(X=1) + P(X=2) + P(X=3) = \frac{2(1)+1}{25} + \frac{2(2)+1}{25} + \frac{2(3)+1}{25} = \frac{15}{25} = \frac{3}{5}$

$P(X > -2) = 1$

$P(X \le 1) = P(X=0) + P(X=1) = \frac{1}{25} + \frac{3}{25} = \frac{4}{25}$

(b) [1 point] Find the cumulative distribution function $F(x)$.

$P(X \le x) = \sum_{i=0}^{x} P(X=i) = \frac{1}{25} \sum_{i=0}^{x} (2i+1)$

$= \frac{1}{25}\left(2\sum_{i=0}^{x} i + \sum_{i=0}^{x} 1\right) = \frac{1}{25} \cdot \left(2\,\frac{x(x+1)}{2} + x+1\right)$

$= \frac{1}{25}(x+1)^2, \quad x = 0, 1, 2, 3, 4$

$$F(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{25}(\lfloor x+1 \rfloor)^2 & 0 \le x \le 4 \\ 1, & x > 4 \end{cases}$$

x	$F(x) = P(X \le x)$
$(-\infty, 0)$	0
$[0,1)$	1/25
$[1,2)$	4/25
$[2,3)$	9/25
$[3,4)$	16/25
$[4,\infty)$	1

(c) [2 points] Find the expectation and the variance of $Y = X\sqrt{2} - 1$.

$E[Y] = E[\sqrt{a}X + b] = \sqrt{a}E[X] + b \Rightarrow 2.8\sqrt{a} + b$

$Var[Y] = Var[\sqrt{a}X + b] = a\,Var[X] \Rightarrow 1.36\,a$

$E[X] = \sum_{x=0}^{4} x\,P(X=x) = 0 \cdot \left(\frac{1}{25}\right) + 1 \cdot \left(\frac{3}{25}\right) + 2 \cdot \left(\frac{5}{25}\right) + 3 \cdot \left(\frac{7}{25}\right) + 4 \cdot \left(\frac{9}{25}\right) = 2.8$

$E[X^2] = \sum_{x=0}^{4} x^2 P(X=x) = 0^2\left(\frac{1}{25}\right) + 1^2\left(\frac{3}{25}\right) + 2^2\left(\frac{5}{25}\right) + 3^2\left(\frac{7}{25}\right) + 4^2\left(\frac{9}{25}\right) = 9.2$

V1. $Var[X] = E[X^2] - E^2[X] = 9.2 - 2.8^2 = 1.36$

$E[Y] = 2.8\sqrt{2} - 1 \cong 2.96$

$V[Y] = 1.36(\sqrt{2})^2 = 2.72$

V2. $E[Y] = 2.8\sqrt{3} + 1 \cong 5.85$

$V[Y] = 1.36(\sqrt{3})^2 = 4.08$

8. The compressive strength of samples of cement can be modeled by a normal distribution with a mean of 6050 kg/cm^2 and a standard deviation of 110 kg/cm^2.

   (a) [1 point] What is the probability that the sample strength is less than 6200 kg/cm^2?

   $X=$ compressive strength , $X \sim N(6050, 110^2)$, $Z = \frac{X-6050}{110} \sim N(0,1)$

   $P(X<6200) = P\left(\frac{X-6050}{110} < \frac{6200-6050}{110}\right) \cong P(Z<1.36)$

   $= \Phi(1.36) \cong 0.9131$

   (b) [1 point] What is the probability that a sample's strength is between 5800 and 6000 kg/cm^2?

   $P(5800<X<6000) = P\left(\frac{5800-6050}{110} < Z < \frac{6000-6050}{110}\right)$

   $\cong P(-2.27 < Z < -0.45) = \Phi(-0.45) - \Phi(-2.27)$

   $= 0.3264 - 0.0116 \cong 0.31$

   (c) [1 point] What strength x_0 (in kg/cm^2) is exceeded by 95% of the samples? (In other words, 95% of the samples have a strength that is larger than x_0.)

   $0.95 = P(X>x_0) \Rightarrow 0.95 = 1 - P(X<x_0)$

   $\Rightarrow P(X<x_0) = 0.05$

   $Z_0 \Rightarrow -1.645 \Rightarrow$

   $X_0 = 110 Z_0 + 6050 \Rightarrow 110(-1.645) + 6050 \cong 5869.05$

   (d) [1 point] What strength x_1 (in kg/cm^2) is exceeded by 10% of the samples?

   $0.1 = P(X>x_1) \Rightarrow 0.1 = 1 - P(X<x_0)$

   $P(X<x_1) = 0.9$

   $Z_1 = 1.285$

   $X_1 = 110 Z_1 + 6050 \Rightarrow 110(1.285) + 6050 \cong 6191.35$

9. [3 points] Amongst all vehicles that have an ignition problem, assume that

- the starter is at fault in 50% of the cases;
- the battery is at fault in 40% of the cases, and
- one of the plugs is at fault the rest of the time.

Furthermore, amongst all vehicles with an ignition problem, the proportion of vehicles of make X is

- 10% in cases where the starter is at fault;
- 20% in cases where the battery is at fault, and
- 5% in cases where one of the plugs is at fault.

A vehicle of make X is brought to a mechanic because of an ignition problem. What should be the first line of attack for the mechanic: starter, battery, or plugs?

Answer:

S: starter at fault
B: battery at fault
L: a plug at fault

	V1	V2
	$P(S)=0.5$	$P(S)=0.6$
	$P(B)=0.4$	$P(B)=0.3$
	$P(L)=0.1$	$P(L)=0.1$
	$P(X\|S)=0.1$	$P(X\|S)=0.05$
	$P(X\|B)=0.2$	$P(X\|B)=0.2$
	$P(X\|L)=0.05$	$P(X\|L)=0.1$

We are interested in $P(S|X)$, $P(B|X)$, $P(L|X)$.

$$P(S|X) = \frac{P(X|S)\,P(S)}{P(X)} \propto \begin{cases} (0.1)(0.5) = 0.05 & V1 \\ (0.05)(0.6) = 0.03 & V2 \end{cases}$$

$$P(B|X) = \frac{P(X|B)\,P(B)}{P(X)} \propto \begin{cases} (0.2)(0.4) = 0.08 & V1 \\ (0.2)(0.3) = 0.06 & V2 \end{cases}$$

$$P(L|X) = \frac{P(X|L)\,P(L)}{P(X)} \propto \begin{cases} (0.05)(0.1) = 0.005 & V1 \\ (0.1)(0.1) = 0.01 & V2 \end{cases}$$

	V1	V2
$P(S\|X)$	$0.05/P(X)$	$0.03/P(X)$
$P(B\|X)$	$0.08/P(X)$	$0.06/P(X)$
$P(L\|X)$	$0.005/P(X)$	$0.01/P(X)$

$\Longrightarrow$ highest probability: check the battery first.

10. [3 points] An electrical system consist of 30 integrated independent circuits. For each of these circuit, the probability of being defective is 0.015. The system is operational if at most 2 circuits are defective. What is the probability that the system is operational?

Answer:

X: # of defective circuits

$X \sim B(30, 0.015)$

$$P(X \le 2) = P(X=0) + P(X=1) + P(X=2)$$

$$= \binom{30}{0}(0.015)^0(0.985)^{30}$$

$$+ \binom{30}{1}(0.015)^1(0.985)^{29}$$

$$+ \binom{30}{2}(0.015)^2(0.985)^{28}$$

$$= (0.985)^{30} + 30(0.015)(0.985)^{29} + \frac{30 \cdot 29}{2}(0.015)^2(0.985)^{28}$$

$$\simeq 0.6355 + 0.2903 + 0.0641$$

$$= 0.9899$$