# MAT 2377
# Probability and Statistics for Engineers

## Chapter 2
## Discrete Distributions

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Based on course notes by Rafał Kulik

# Contents

# 2.1 – Random Variables and Distributions

Recall that, for any random "experiment," the set of all possible outcomes is denoted by $\mathcal{S}$.

A **random variable** (r.v.) is a function $X : \mathcal{S} \to \mathbb{R}$, i.e. it is a rule that associates a (real) number to every outcomes of the experiment.

$\mathcal{S}$ is the **domain** of the r.v. $X$; $X(\mathcal{S}) \subseteq \mathbb{R}$ is its **range**.

A **probability distribution function** (p.d.f.) is a function $f : \mathbb{R} \to \mathbb{R}$ which specifies the probabilities of the values in $X(\mathcal{S})$.

When $\mathcal{S}$ is discrete, we say that $X$ is a **discrete r.v.** and the p.d.f. is called a **probability mass function** (p.m.f.).

# Notation for R.V.

We use the following notation throughout:

- capital roman letters ($X$, $Y$, etc.) to denote r.v.
- corresponding lower case roman letters ($x$, $y$, etc.) to denote $generic$ $values$ $taken$ by the r.v.

A discrete r.v. can be used to define events: if $X$ takes values $X(\mathcal{S}) = \{x_i\}$, then we can define events $A_i = \{s \in \mathcal{S} : X(s) = x_i\}$.

- The p.m.f. of $X$ is $f(x) = P(\{s \in \mathcal{S} : X(s) = x\}) := P(X = x)$.
- The cumulative distribution function (c.d.f.) of $X$ is $F(x) = P(X \leq x)$.

# Properties of the P.M.F. and the C.D.F.

If $X$ is a discrete random variable with p.m.f. $f(x)$ and c.d.f. $F(x)$, then

- $0 < f(x) \leq 1$ for all $x \in X(\mathcal{S})$;
- $\sum_{s \in \mathcal{S}} f(X(s)) = \sum_{x \in X(\mathcal{S})} f(x) = 1$;
- for any event $A \subseteq \mathcal{S}$, $P(X \in A) = \sum_{x \in A} f(x)$;
- for any $a, b \in \mathbb{R}$,

$$P(a < X) = 1 - P(X \leq a) = 1 - F(a)$$
$$P(X < b) = P(X \leq b) - P(X = b) = F(b) - f(b)$$

- for any $a, b \in \mathbb{R}$,

$$\begin{aligned} P(a \leq X) &= 1 - P(X < a) = 1 - (P(X \leq a) - P(X = a)) \\ &= 1 - F(a) + f(a) \end{aligned}$$

We can use these results to compute the probability of a **discrete** r.v. $X$ falling in various intervals:

$$P(a < X \leq b) = P(X \leq b) - P(X \leq a) = F(b) - F(a)$$

$$P(a \leq X \leq b) = P(a < X \leq b) + P(X = a) = F(b) - F(a) + f(a)$$

$$P(a < X < b) = P(a < X \leq b) - P(X = b) = F(b) - F(a) - f(b)$$

$$P(a \leq X < b) = P(a \leq X \leq b) - P(X = b) = F(b) - F(a) + f(a) - f(b)$$

## Examples:

1. Flip a fair coin. The outcome space is $\mathcal{S} = \{\text{Head}, \text{Tail}\}$. Let $X : S \to \mathbb{R}$ be defined by $X(\text{Head}) = 1$ and $X(\text{Tail}) = 0$. Then $X$ is a discrete random variable (as a convenience, we write $X = 1$ and $X = 0$).

   If the coin is fair, the p.m.f. of $X$ is $f : \mathbb{R} \to \mathbb{R}$, where

   $$f(0) = P(X = 0) = 1/2,\ f(1) = P(X = 1) = 1/2,\ f(x) = 0 \text{ for all other } x.$$

2. Roll a fair die. The outcome space is $\mathcal{S} = \{1, \ldots, 6\}$. Let $X : \mathcal{S} \to \mathbb{R}$ be defined by $X(i) = i$ for $i = 1, \ldots, 6$. Then $X$ is a discrete r.v.

   If the die is fair, the p.m.f. of $X$ is $f : \mathbb{R} \to \mathbb{R}$, where

   $$f(i) = P(X = i) = 1/6, \text{ for } i = 1, \ldots, 6,\ f(x) = 0 \text{ for all other } x.$$

3. For the random variable $X$ from the previous example, the c.d.f. is $F : \mathbb{R} \to \mathbb{R}$, where

$$F(x) = P(X \leq x) = \begin{cases} 0 & \text{if } x < 1 \\ i/6 & \text{if } i \leq x < i+1, \text{ for } i = 1, \ldots, 6 \\ 1 & \text{if } x \geq 6 \end{cases}$$

4. For the same random variable, we can compute $P(3 \leq X \leq 5)$ directly:

$$P(3 \leq X \leq 5) = P(X = 3) + P(X = 4) + P(X = 5) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2},$$

or we can use the c.d.f.

$$P(3 \leq X \leq 5) = F(5) - F(3) + f(3) = \frac{5}{6} - \frac{3}{6} + \frac{1}{6}$$

5. The number of calls received over a specific time period, $X$, is a discrete random variable, with potential values $0, 1, 2, \ldots$.

6. Consider a $5-$card poker hand consisting of cards selected at random from a $52-$card deck. Find the probability distribution of $X$, where $X$ indicates the number of red cards ($\diamondsuit$ and $\heartsuit$) in the hand.

   **Solution:** in all there are $\binom{52}{5}$ ways to select a $5-$card poker hand from a $52-$card deck.

   By construction, $X$ can take on values $x = 0, 1, 2, 3, 4, 5$.

If $X = 0$, then none of the $5$ cards in the hands are $\diamondsuit$ or $\heartsuit$, and all of the $5$ cards in the hands are $\spadesuit$ or $\clubsuit$.

There are thus $\binom{26}{0} \cdot \binom{26}{5}$ $5-$card hands that only contain black cards, and

$$P(X = 0) = \frac{\binom{26}{0} \cdot \binom{26}{5}}{\binom{52}{5}}.$$

In general, if $X = x$, $x = 0, 1, 2, 3, 4, 5$, there are $\binom{26}{x}$ ways of having $x$ $\diamondsuit$ or $\heartsuit$ in the hand, and $\binom{26}{5-x}$ ways of having $5 - x$ $\spadesuit$ and $\clubsuit$ in the hand, so that

$$f(x) = P(X = x) = \frac{\binom{26}{x} \cdot \binom{26}{5-x}}{\binom{52}{5}}, \quad x = 0, 1, 2, 3, 4, 5;\ f(x) = 0 \text{ otherwise.}$$

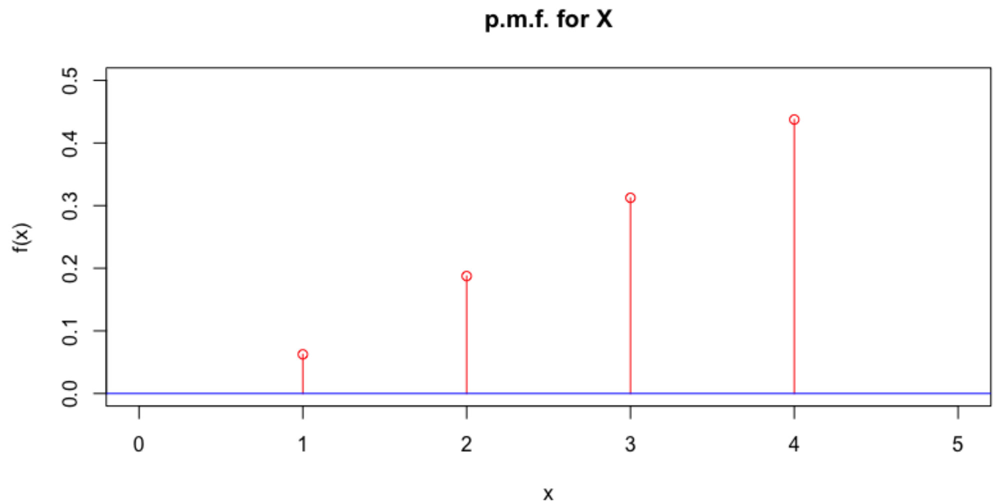
7. Find the c.d.f. $F(x)$ of a discrete random variable $X$ with p.m.f. $f(x)$ defined by $f(x) = 0.1x$ if $x = 1, 2, 3, 4$ and $f(x) = 0$ otherwise.

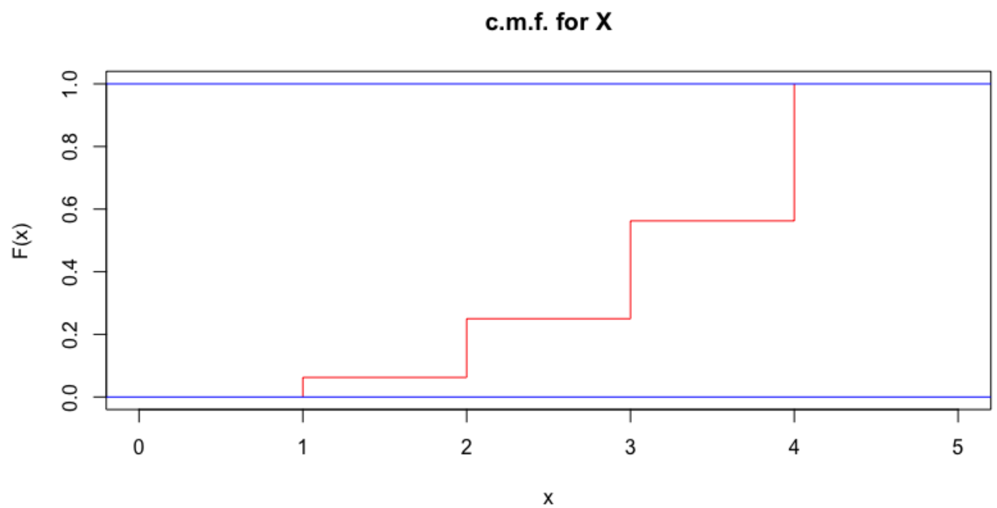
**Solution:** $f(x)$ is indeed a p.m.f. because $0 < f(x) \leq 1$ for all $x$ and

$$\sum_{x=1}^{4} 0.1x = 0.1(1 + 2 + 3 + 4) = 0.1\frac{4(5)}{2} = 1.$$

We have

$$F(x) = \begin{cases} 0 & \text{if } x < 1 \\ 0.1 & \text{if } 1 \leq x < 2 \\ 0.3 & \text{if } 2 \leq x < 3 \\ 0.6 & \text{if } 3 \leq x < 4 \\ 1 & \text{if } x \geq 4 \end{cases}$$


p.m.f. for X
f(x)
0.0
0.1
0.2
0.3
0.4
0.5
0
1
2
3
4
5
x


c.m.f. for X
F(x)
0.0
0.2
0.4
0.6
0.8
1.0
0
1
2
3
4
5
x

# 2.2 – Expectation of a Discrete R.V.

The **expectation** of a discrete random variable $X$ is defined as

$$\mathrm{E}[X] = \sum_x x \cdot P(X = x) = \sum_x x f(x)\,,$$

where the sum extends over all values of $x$ taken by $X$.

The definition can be extended to a general function of $X$:

$$\mathrm{E}[u(X)] = \sum_x u(x) P(X = x) = \sum_x u(x) f(x).$$

As an important example, $\mathrm{E}[X^2] = \sum_x x^2 P(X = x) = \sum_x x^2 f(x)$.

## Examples:

1. What is the expectation on the roll $Z$ of fair $6-$sided die?

   **Solution:** $\mathrm{E}[Z] = \sum_{z=1}^{6} z \cdot P(Z = z) = \frac{1}{6} \sum_{z=1}^{6} z = \frac{1}{6} \cdot \frac{6(7)}{2} = 3.5.$

2. For each $1\$$ bet in a gambling game, a player can win $3\$$ with probability $\frac{1}{3}$ and lose $1\$$ with probability $\frac{2}{3}$. Let $X$ be the net gain/loss from the game. Find the expected value of the game.

   **Solution:** $X$ can take on the value $2\$$ (for a win) and $-2\$$ for a loss (outcome $-$ bet). The expected value of $X$ is thus

$$\mathrm{E}[X] = 2 \cdot \frac{1}{3} + (-2) \cdot \frac{2}{3} = -\frac{2}{3}.$$

3. If $Z$ is the number showing on a roll of a fair $6-$sided die, find $\mathrm{E}[Z^2]$ and $\mathrm{E}[(Z-3.5)^2]$.

**Solution:**

$$\mathrm{E}[Z^2] = \sum_z z^2 P(Z=z) = \frac{1}{6}\sum_{z=1}^{6} z^2 = \frac{1}{6}(1^2+\cdots+6^2) = \frac{91}{6}$$

$$\begin{aligned}\mathrm{E}[(Z-3.5)^2] &= \sum_{z=1}^{6}(z-3.5)^2 P(Z=z) = \frac{1}{6}\sum_{z=1}^{6}(z-3.5)^2 \\ &= \frac{(1-3.5)^2+\cdots+(6-3.5)^2}{6} = \frac{35}{12}\end{aligned}$$

# Mean and Variance of a Discrete R.V.

We can interpret the expectation as the average or the **mean** of $X$, which we often denote by $\mu = \mu_X$.

For instance, in the example of the fair die, $\mu_Z = \mathrm{E}[Z] = 3.5$.

Note that in the final example, we could have written

$$\mathrm{E}[(Z - 3.5)^2] = \mathrm{E}[(Z - \mathrm{E}[Z])^2].$$

This is an important quantity associated to a random variable $X$, its **variance** $\mathrm{Var}[X]$.

The variance of a discrete random variable $X$ is the **expected squared difference from the mean**:

$$\begin{aligned}\mathrm{Var}(X) = \mathrm{E}[(X-\mu_X)^2] &= \sum_x (x-\mu_X)^2 P(X=x) \\ &= \sum_x \left(x^2 - 2x\mu_X + \mu_X^2\right) f(x) \\ &= \sum_x x^2 f(x) - 2\mu_X \sum_x x f(x) + \mu_X^2 \sum_x f(x) \\ &= \mathrm{E}[X^2] - 2\mu_X\mu_X + \mu_X^2 \cdot 1 = \mathrm{E}[X^2] - \mu_X^2.\end{aligned}$$

This is also sometimes written as $\mathrm{Var}[X] = \mathrm{E}[X^2] - \mathrm{E}^2[X]$.

# Standard Deviation

The **standard deviation** of a discrete random variable $X$ is defined directly from the variance:

$$\mathrm{SD}[X] = \sqrt{\mathrm{Var}[X]}\,.$$

The mean gives some idea as to where the **bulk** of a distribution is located $\Rightarrow$ measure of **centrality** (more on this later).

The variance and standard deviation provide information about the **spread**; distributions with higher variance/SD are **more spread out about the average**.

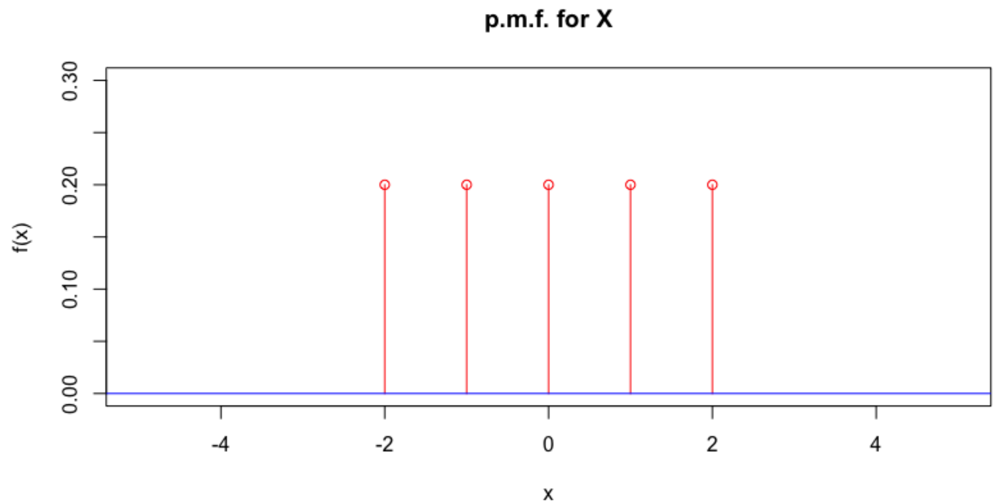
**Examples:** let $X$ and $Y$ be random variables with the following p.d.f.:

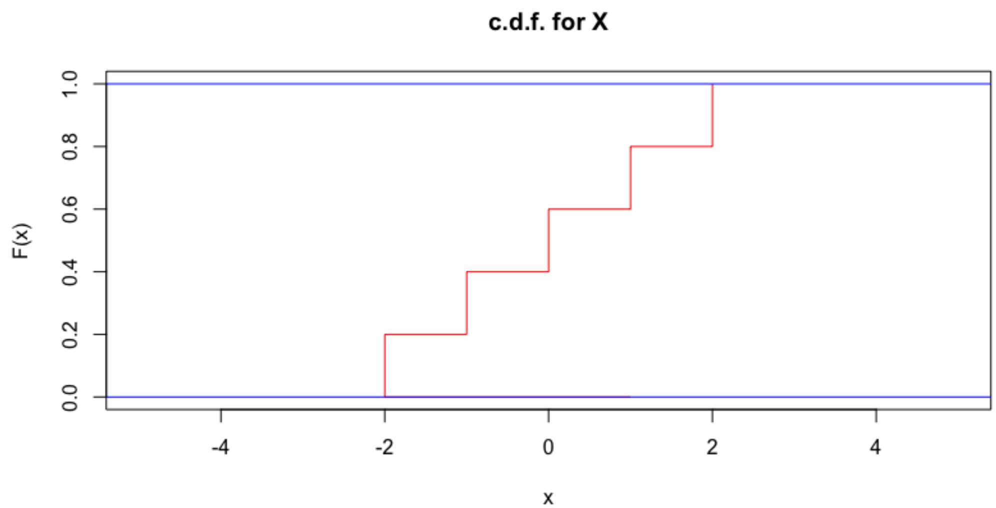
| $x$ | $P(X = x)$ |
|---|---|
| $-2$ | $1/5$ |
| $-1$ | $1/5$ |
| $0$ | $1/5$ |
| $1$ | $1/5$ |
| $2$ | $1/5$ |

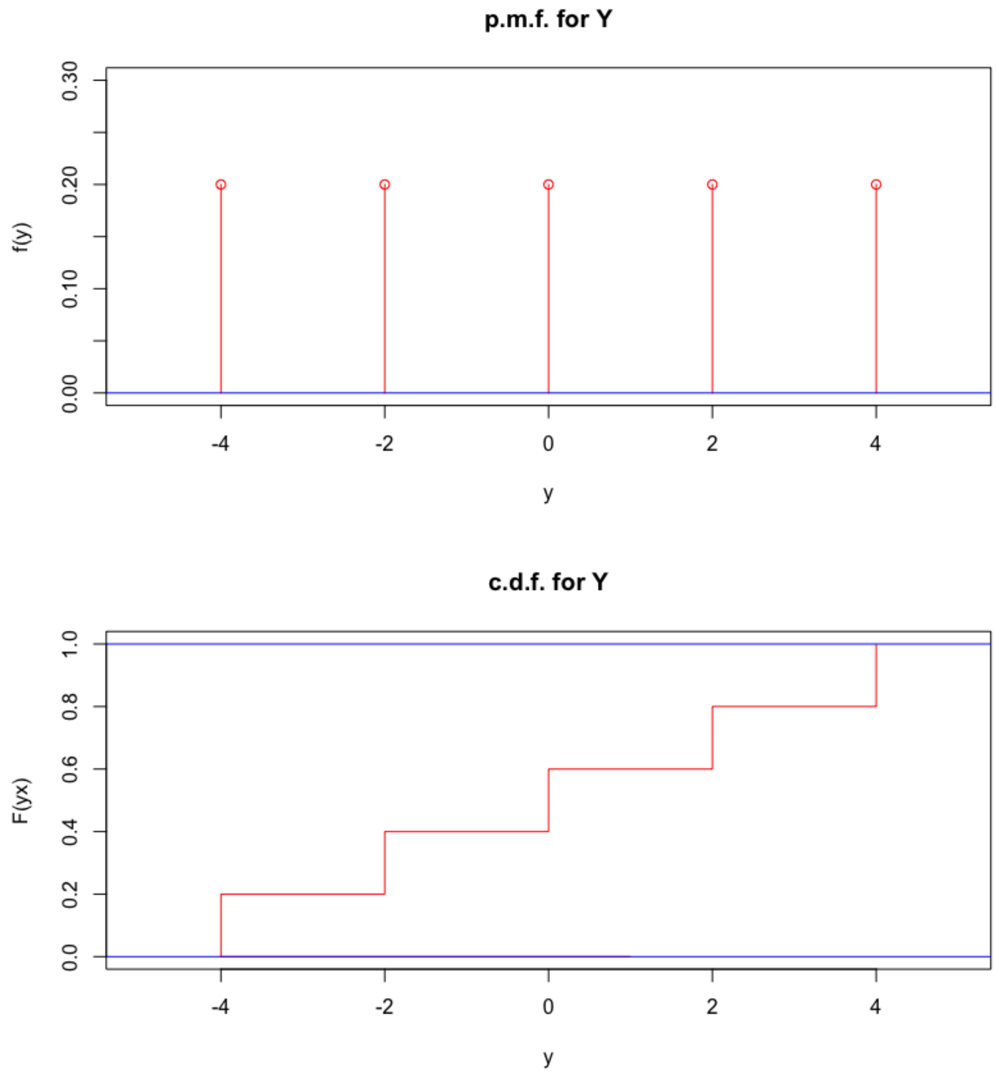
| $y$ | $P(Y = y)$ |
|---|---|
| $-4$ | $1/5$ |
| $-2$ | $1/5$ |
| $0$ | $1/5$ |
| $2$ | $1/5$ |
| $4$ | $1/5$ |

Compute the expected values and compare the variances.

**Solution:** We have $\mathrm{E}[X] = \mathrm{E}[Y] = 0$ and $2 = \mathrm{Var}[X] < \mathrm{Var}[Y] = 8$, meaning that we would expect both distributions to be centered at $0$, but $Y$ should be more spread-out than $X$.


p.m.f. for X
f(x)
0.00
0.10
0.20
0.30
-4
-2
0
2
4
x


c.d.f. for X
F(x)
0.0
0.2
0.4
0.6
0.8
1.0
-4
-2
0
2
4
x


p.m.f. for Y
f(y)
0.00
0.10
0.20
0.30
-4
-2
0
2
4
y
c.d.f. for Y
F(yx)
0.0
0.2
0.4
0.6
0.8
1.0
-4
-2
0
2
4
y

# Properties of Expectations

For all $a \in \mathbb{R}$:

- $\mathrm{E}[aX] = a\mathrm{E}[X]$;
- $\mathrm{E}[X + a] = \mathrm{E}[X] + a$;
- $\mathrm{E}[X + Y] = \mathrm{E}[X] + \mathrm{E}[Y]$;
- In general, $\mathrm{E}[XY] \neq \mathrm{E}[X]\mathrm{E}[Y]$;
- $\mathrm{Var}[aX] = a^2\mathrm{Var}[X]$, $\mathrm{SD}[aX] = |a|\mathrm{SD}[X]$;
- $\mathrm{Var}[X + a] = \mathrm{Var}[X]$, $\mathrm{SD}[X + a] = \mathrm{SD}[X]$.

# 2.3 – Binomial Distribution

Recall the number of unordered samples of size $r$ from a set of size $n$ is:

$$_nC_r = \binom{n}{r} = \frac{n!}{(n-r)!r!}.$$

- $2! \times 4! = (1 \times 2) \times (1 \times 2 \times 3 \times 4) = 48$, but $(2 \times 4)! = 8! = 40320.$
- $\binom{5}{1} = \frac{5!}{1! \times 4!} = \frac{1 \times 2 \times 3 \times 4 \times 5}{1 \times (1 \times 2 \times 3 \times 4)} = \frac{5}{1} = 5$; In general: $\binom{n}{1} = n$. Also $\binom{n}{0} = 1$
- $\binom{6}{2} = \frac{6!}{2! \times 4!} = \frac{4! \times 5 \times 6}{2! \times 4!} = \frac{5 \times 6}{2} = 15$
- $\binom{27}{22} = \frac{27!}{22! \times 5!} = \frac{22! \times 23 \times 24 \times 25 \times 26 \times 27}{5! \times 22!} = \frac{23 \times 24 \times 25 \times 26 \times 27}{120}$

# Binomial Experiments

A **Bernoulli trial** is a random experiment with two possible outcomes, “success” and “failure”. Let $p$ denote the probability of a success.

A **binomial experiment** consists of $n$ repeated independent Bernoulli trials, each with the same probability of success, $p$.

**Examples:**

- female/male births;
- satisfactory/defective items on a production line;
- sampling with replacement with two types of item, etc.

# Probability Mass Function

In a binomial experiment of $n$ independent events, each with probability of success $p$, the number of successes $X$ is a discrete random variable that follows a **binomial distribution** with parameters $(n, p)$:

$$f(x) = P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}\,, \quad \text{for } x = 0, 1, 2, \ldots, n.$$

This is often abbreviated to "$X \sim \mathcal{B}(n, p)$".

If $X \sim B(1, p)$ then $P(X = 0) = 1 - p$, $P(X = 1) = p$, and so

$$\mathrm{E}[X] = (1-p) \cdot 0 + p \cdot 1 = p\,.$$

# Expectation and Variance for $\mathcal{B}(n, p)$

If $X \sim \mathcal{B}(n, p)$, then $P(X = x)$ is as on the previous slide and it can be shown that

$$\mathrm{E}[X] = \sum_{x=0}^{n} xP(X = x) = np,$$

and

$$\mathrm{Var}[X] = \mathrm{E}\left[(X - np)^2\right] = \sum_{x=0}^{n}(x - np)^2 P(X = x) = np(1 - p)\,.$$

Later we will see an easier way to derive these by interpreting $X$ as a sum of other discrete random variables.

## Examples:

1. Suppose that each water sample taken in some well-defined region has a $10\%$ probability of being polluted.

   If $12$ samples are selected independently, then it is reasonable to model the number $X$ of polluted samples as $\mathcal{B}(12, 0.1)$. Find

   a) $\mathrm{E}[X]$ and $\mathrm{Var}[X]$;
   b) $P(X = 3)$;
   c) $P(X \leq 3)$.

   **Solution**:

   a) If $X \sim \mathcal{B}(n, p)$ then $\mathrm{E}[X] = np$ and $\mathrm{Var}[X] = np(1 - p)$, so

$$\mathrm{E}[X] = 12 \times 0.1 = 1.2; \quad \mathrm{Var}[X] = 12 \times 0.1 \times 0.9 = 1.08\,.$$

b) By definition, $P(X = 3) = \binom{12}{3}(0.1)^3(0.9)^9 \approx 0.0852$.

c) By definition,

$$P(X \leq 3) = P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) = \cdots$$

However, for $X \sim \mathcal{B}(12, 0.1)$, $P(X \leq 3)$ is tabulated on p.430 of text (Table A.1), and is $\approx 0.9744$.

The table can also be used to compute

$$P(X = 3) = P(X \leq 3) - P(X \leq 2) = 0.9744 - 0.8891 \approx 0.0853.$$

Note the rounding error.

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 12 | 0 | 0.2824 | 0.0687 | 0.0138 | 0.0022 | 0.0002 | 0.0000 | | | |
| | 1 | 0.6590 | 0.2749 | 0.0850 | 0.0196 | 0.0032 | 0.0003 | 0.0000 | | |
| | 2 | 0.8891 | 0.5583 | 0.2528 | 0.0834 | 0.0193 | 0.0028 | 0.0002 | | |
| | 3 | 0.9744 | 0.7946 | 0.4925 | 0.2253 | 0.0730 | 0.0153 | 0.0017 | 0.0000 | |
| | 4 | 0.9957 | 0.9274 | 0.7237 | 0.4382 | 0.1938 | 0.0573 | 0.0095 | 0.0006 | |
| | 5 | 0.9995 | 0.9806 | 0.8822 | 0.6652 | 0.3872 | 0.1582 | 0.0386 | 0.0039 | 0.0000 |
| | 6 | 0.9999 | 0.9961 | 0.9614 | 0.8418 | 0.6128 | 0.3348 | 0.1178 | 0.0194 | 0.0005 |
| | 7 | 1.0000 | 0.9994 | 0.9905 | 0.9427 | 0.8062 | 0.5618 | 0.2763 | 0.0726 | 0.0043 |
| | 8 | | 0.9999 | 0.9983 | 0.9847 | 0.9270 | 0.7747 | 0.5075 | 0.2054 | 0.0256 |
| | 9 | | 1.0000 | 0.9998 | 0.9972 | 0.9807 | 0.9166 | 0.7472 | 0.4417 | 0.1109 |
| | 10 | | | 1.0000 | 0.9997 | 0.9968 | 0.9804 | 0.9150 | 0.7251 | 0.3410 |
| | 11 | | | | 1.0000 | 0.9998 | 0.9978 | 0.9862 | 0.9313 | 0.7176 |
| | 12 | | | | | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 |

Table of c.d.f. $F(x) = P(X \leq x)$ for $X \sim \mathcal{B}(12, p)$, $p = 0.1, \ldots, 0.9$.

2. An airline sells $101$ tickets for a flight with $100$ seats. Each passenger with a ticket is known to have a $p = 0.97$ probability of showing up for their flight. What is the probability of $101$ passengers showing up (and the airline being caught overbooking)? Make appropriate assumptions. What if the airline sells 125 tickets?

   **Solution:** let $X$ be the number of passengers that show up. We want to compute $P(X > 100)$.

   If all passengers show up independently of one another (no families or late bus?), we can model $X \sim \mathcal{B}(101, 0.97)$ and

$$P(X > 100) = P(X = 101) = \binom{101}{101}(0.97)^{101}(0.03)^0 \approx 0.046$$

If the airline sells $n = 125$ tickets, we can model $X \sim \mathcal{B}(125, 0.97)$ and

$$P(X > 100) = 1 - P(X \leq 100) = 1 - \sum_{x=0}^{100} \binom{125}{x} (0.97)^x (0.03)^{125-x}.$$

This is harder to compute directly, but is very nearly $1$ (try it in `R`).

# 2.4 – Geometric Distribution

Consider a sequence of Bernoulli trials, with probability $p$ of success at each step.

Let the **geometric** random variable $X$ denote the number of steps before the first success occurs. The probability distribution is given by

$$f(x) = P(X = x) = (1-p)^{x-1}p, \quad x = 1, \ldots$$

We will write $X \sim \text{Geo}(p)$. For this random variable, we have

$$\mathrm{E}[X] = \frac{1}{p} \quad \text{and} \quad \mathrm{Var}[X] = \frac{1-p}{p^2}.$$

## Examples:

- A fair $6-$sided die is thrown until it shows a $6$. What is the probability that $5$ throws are required?

  **Solution:** If $5$ throws are required, we have to compute $P(X = 5)$, where $X$ is geometric Geo$(1/6)$:

$$P(X = 5) = (1 - p)^{5-1}p = (5/6)^4(1/6) \approx 0.0804.$$

- In the example above, how many throws would you expect to need?

  **Solution:** $\mathrm{E}[X] = \frac{1}{1/6} = 6$.

# 2.5 – Negative Binomial Distribution

Consider a sequence of Bernoulli trials, with probability $p$ of success at each step.

Let the **negative binomial** random variable $X$ denote the number of steps before the $r$th success occurs. The probability distribution is given by

$$f(x) = P(X = x) = \binom{x-1}{r-1}(1-p)^{x-r}p^r, \quad x = r, \ldots$$

We will write $X \sim \text{NegBin}(p, r)$. For this random variable, we have

$$\text{E}[X] = \frac{r}{p} \quad \text{and} \quad \text{Var}[X] = \frac{r(1-p)}{p^2}.$$

## Examples:

- A fair $6-$sided die is thrown until it three $6$'s are rolled. What is the probability that $5$ throws are required?

  **Solution:** If $5$ throws are required, we have to compute $P(X = 5)$, where $X$ is geometric NegBin$(1/6, 3)$:

$$P(X = 5) = \binom{5-1}{3-1}(1-p)^{5-3}p^3 = \binom{4}{2}(5/6)^2(1/6)^3 \approx 0.0193.$$

- In the example above, how many throws would you expect to need?

  **Solution:** $\mathrm{E}[X] = \frac{3}{1/6} = 18.$

# 2.6 – Poisson Distribution

We count the number of "changes" that occur in a continuous interval of time or space (such as # of defects on a production line over a $1$ hr period, # of customers that arrive at a teller over a $15$ min interval, etc.).

We have a **Poisson process** with rate $\lambda$, denoted by $\mathcal{P}(\lambda)$, if:

a) the number of changes occurring in non-overlapping intervals are independent;

b) the probability of exactly one change in a short interval of length $h$ is approximately $\lambda h$, and

c) The probability of $2+$ changes in a sufficiently short interval is essentially $0$.

Assume that an experiment satisfies the above properties. Let $X$ be the number of changes in a **unit interval** (this could be $1$ day, or $15$ minutes, or $10$ years, etc.).

What is $P(X = x)$, for $x = 0, 1, \ldots$?

Partition the unit interval into $n$ disjoint sub-intervals of length $1/n$.

1. By condition b), the probability of one change occurring in one of the sub-intervals is approximately $\lambda/n$.

2. By condition c), the probability of $2+$ changes is $\approx 0$.

3. By condition a), we have a sequence of $n$ Bernoulli trials with probability $p = \lambda/n$.

Therefore,

$$f(x) = P(X = x) \approx \frac{n!}{x!(n-x)!}\left(\frac{\lambda}{n}\right)^x \left(1 - \frac{\lambda}{n}\right)^{n-x}$$

$$= \frac{\lambda^x}{x!} \cdot \underbrace{\frac{n!}{(n-x)!} \cdot \frac{1}{n^x}}_{\text{term 1}} \cdot \underbrace{\left(1 - \frac{\lambda}{n}\right)^n}_{\text{term 2}} \cdot \underbrace{\left(1 - \frac{\lambda}{n}\right)^{-x}}_{\text{term 3}}.$$

Letting $n \to \infty$, we get

$$P(X = x) = \lim_{n\to\infty} \frac{\lambda^x}{x!} \cdot \underbrace{\frac{n!}{(n-x)!} \cdot \frac{1}{n^x}}_{\text{term 1}} \cdot \underbrace{\left(1 - \frac{\lambda}{n}\right)^n}_{\text{term 2}} \cdot \underbrace{\left(1 - \frac{\lambda}{n}\right)^{-x}}_{\text{term 3}}$$

$$= \frac{\lambda^x}{x!} \cdot 1 \cdot \exp(-\lambda) \cdot 1 = \frac{\lambda^x e^{-\lambda}}{x!}, \quad x = 0, 1, \ldots$$

Let $X \sim \mathcal{P}(\lambda)$. Then it can be shown that

$$\mathrm{E}[X] = \lambda \quad \text{and} \quad \mathrm{Var}[X] = \lambda,$$

that is, the mean and the variance of a Poisson random variable are identical!

**Examples:**

1. A traffic flow is typically modeled by a Poisson distribution. It is known that the traffic flowing through an intersection is $6$ cars/minute, on average. What is the probability of no cars entering the intersection in a $30$ second period?

   **Solution:** $6$ cars/min $=$ $3$ cars/$30$ sec. Thus $\lambda = 3$, and we need to compute

$$P(X = 0) = \frac{3^0 e^{-3}}{3!} = \frac{e^{-3}}{6} \approx 0.0498.$$

2. A hospital needs to schedule night shifts in the maternity ward.

   It is known that there are $3000$ deliveries per year; if these happened randomly round the clock (is this a reasonable assumption?), we would expect $1000$ deliveries between the hours of midnight and 8.00 a.m., a time when much of the staff is off-duty.

   It is thus important to ensure that the night shift is sufficiently staffed to allow the maternity ward to cope with the workload on any particular night, or at least, on a high proportion of nights.

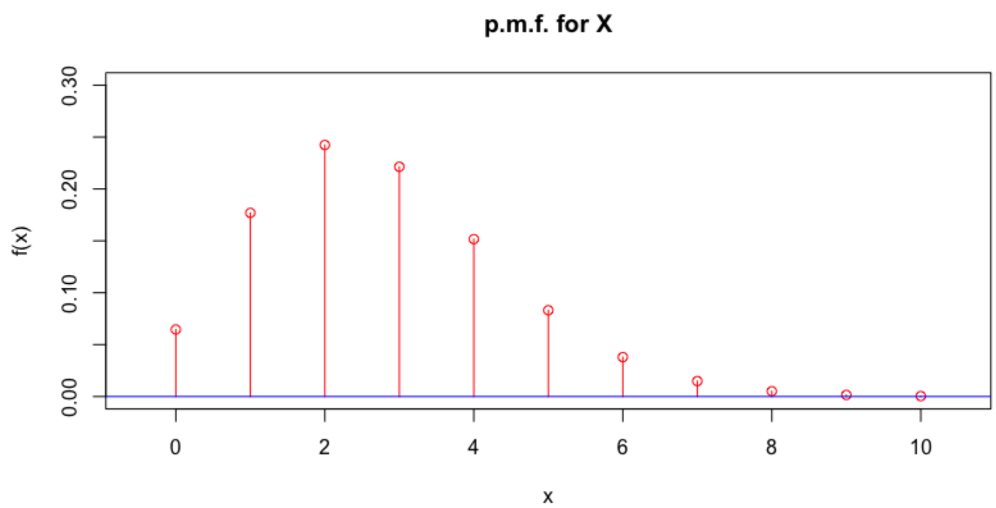
   The average number of deliveries per night is $\lambda = 1000/365.25 \approx 2.74$. If the daily number $X$ of night deliveries follows a Poisson process $\mathcal{P}(\lambda)$, we can compute the probability of delivering $x = 0, 1, 2, \ldots$ babies on each night.

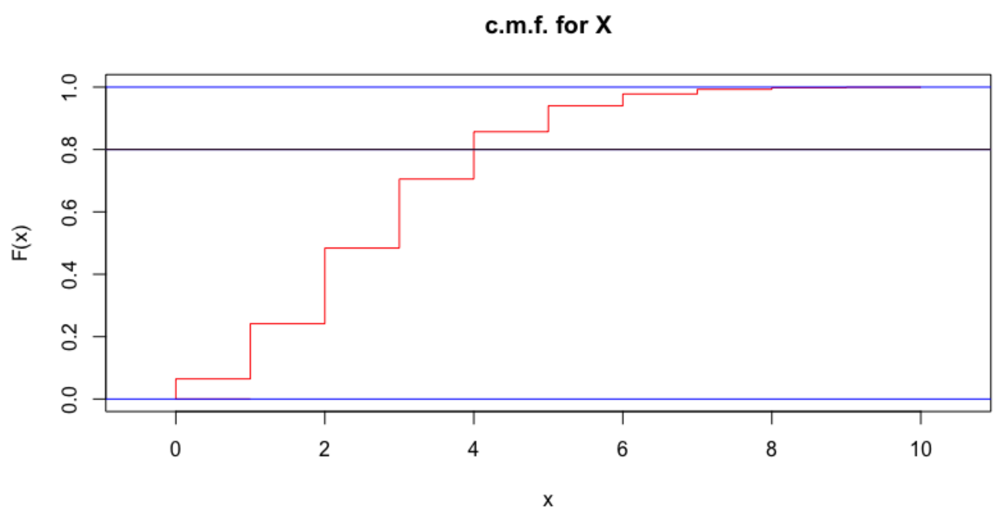
Some of the probabilities are:

| $P(X = x)$ | $\lambda^x \cdot \exp(-\lambda)/x!$ |
|---|---|
| $P(X = 0)$ | $2.74^0 \cdot \exp(-2.74)/0! = 0.065$ |
| $P(X = 1)$ | $2.74^1 \cdot \exp(-2.74)/1! = 0.177$ |
| $P(X = 2)$ | $2.74^2 \cdot \exp(-2.74)/2! = 0.242$ |
| ... | ... |

3. If the maternity ward wants to prepare for the greatest possible traffic on $80\%$ of the nights, how many deliveries should be expected?

   **Solution:** we seek an $x$ for which $P(X \leq x - 1) \leq 0.80 \leq P(X \leq x)$: since `ppois(3,2.74)=.705` and `ppois(4,2.74)=.857`, if they prepare for $4$ deliveries a night, they will be ready for the worst on at least $80\%$ of the nights (closer to $85.7\%$, actually). Note that this is different than asking how many deliveries are expected nightly (namely, $\mathrm{E}[X] = 2.74$).


p.m.f. for X
f(x)
0.00
0.10
0.20
0.30
0
2
4
6
8
10
x


c.m.f. for X
F(x)
0.0
0.2
0.4
0.6
0.8
1.0
0
2
4
6
8
10
x

4. On how many nights in the year would $5$ or more deliveries be expected?

   **Solution:** we need to evaluate

$$365.25 \cdot P(X \geq 5) = 365.25(1 - P(X \leq 4))$$
$$= 365.25 * (1 - \texttt{ppois}(4, 2.74)) \approx 52.27.$$

5. Over the course of one year, what is the greatest number of deliveries expected on any night?

   **Solution:** look for largest value of $x$ for which $365.25 \cdot P(X = x) \geq 1$.

```
> nights=c() # initializing vector
> for(j in 0:10){nights[j+1]=365.25*dpois(j,2.74)}; # p.m.f.
> max(which(nights>1))-1 # identify largest index
```

$\Rightarrow x = 8$.

# Appendix – Summary

| $X$ | **Description** | $P(X = x)$ | **Domain** | $\mathrm{E}[X]$ | $\mathrm{Var}[X]$ |
|---|---|---|---|---|---|
| **Uniform (Discrete)** | Equally likely outcomes | $\frac{1}{b-a+1}$ | $a, \ldots, b$ | $\frac{a+b}{2}$ | $\frac{(b-a+2)(b-a)}{12}$ |
| **Binomial** | Number of successes in $n$ trials | $\binom{n}{x} p^x (1-p)^{n-x}$ | $0, \ldots, n$ | $np$ | $np(1-p)$ |
| **Poisson** | Number of arrivals in a fixed period of time | $\frac{\lambda^x \exp(-\lambda)}{x!}$ | $0, 1, \ldots$ | $\lambda$ | $\lambda$ |

# Summary

| $X$ | **Description** | $P(X=x)$ | **Domain** | $\mathrm{E}[X]$ | $\mathrm{Var}[X]$ |
|---|---|---|---|---|---|
| **Geometric** | Number of trials until $1^{st}$ success | $(1-p)^{x-1}p$ | $1,2,\ldots$ | $\frac{1}{p}$ | $\frac{1-p}{p^2}$ |
| **Negative Binomial** | Number of trials until $k^{th}$ successes | $\binom{x-1}{k-1}(1-p)^{x-k}p^k$ | $k,k+1,\ldots$ | $\frac{k}{p}$ | $\frac{k(1-p)}{p^2}$ |