MAT 2125 – Homework 2 – Solutions

(due at midnight on February 12, in Brightspace)

1 Formality

In the first assignment, we asked you to justify many steps directly from the axioms for $\mathbb{R}$. Starting with this homework, we can be slightly less formal – we can take it that it is understood what the number $\frac{21}{4}$ means for instance, as is the case for more exotic numbers like $3^{\frac{1}{7}}$. We can also freely use "obvious" facts, like the fact that the maximum of a finite collection of real numbers exists and is finite, without directly citing anything.

With this comes some danger. As we will see in Section 4 of this homework, sometimes "obvious"-looking statements are actually false. The only way around this is to use our judgement. Since this is an introductory course, we ask you to err on the side of caution: if you haven't seen something proved, and it looks like the sort of thing that could be an exercise, you should prove it.

2 Limits

1. Suppose that $\{a_n\}$ is a bounded sequence and $\lim\limits_{n\to\infty} b_n = 0$. Show that $\lim\limits_{n\to\infty} a_n b_n = 0$.

   Proof: Since $\{a_n\}$ is bounded, there exists some $0 \leq M < \infty$ so that $\sup_n |a_n| \leq M$. Next, we will check that $a_n b_n \to 0$. Fix some $\epsilon > 0$. Since $\lim_{n\to\infty} b_n = 0$, there exists some N so that for all $n > N$, $|b_n| \leq \frac{\epsilon}{M}$. Thus, for all $n > N$,

   $$|a_n b_n| \leq M|b_n| \leq M\,\frac{\epsilon}{M} = \epsilon.$$

   Thus, $a_n b_n \to 0$. ■

2. Suppose that $\{a_n\}$ is a sequence and $\lim\limits_{n\to\infty} a_n$ exists. Show that $\{a_n\}$ is a bounded sequence.

   Proof: Let $L = \lim_{n\to\infty} a_n$. Since this limit exists, there exists some N so that for all $n > N$, $|a_n - L| \leq 1$. In particular, by the triangle inequality, $|a_n| \leq |L| + 1$. Thus, for all $m \in \mathbb{N}$,

   $$|a_m| \leq \max\left\{\max_{1\leq j\leq N}\{|a_j|\}, |L|+1\right\}.$$

   Since the right-hand side is a maximum of a finite set, it is finite. ■

3. Show that, for all $c \in (0,\infty)$, $\lim\limits_{n\to\infty} c^{\frac{1}{n}} = 1$. **Vague hint:** For $c > 1$, consider the sequence x_n defined by $x_n = (1+a_n)^n$, find a good linear approximation of $(1+a_n)^n$, and apply an earlier part of this question.

   Proof: For $c = 1$ the result is clear. For $c > 1$, define the sequence $a_n = c^{\frac{1}{n}} - 1 \geq 0$, so that $c = (1+a_n)^n$. Since $a_n \geq 0$, we can write

   $$c = (1+a_n)^n = 1 + \binom{n}{1}a_n + \binom{n}{2}a_n^2 + \ldots \geq 1 + na_n.$$

   Rearranging, we find

   $$0 \leq a_n \leq \frac{c-1}{n}.$$

   Thus, by part 1 of this question, $\lim_{n\to\infty} a_n = 0$. But $a_n = c^{\frac{1}{n}} - 1$, so this implies $\lim_{n\to\infty} c^{\frac{1}{n}} = 1$ as desired.

   Finally, when $0 < c < 1$, we can apply the earlier result to see that

   $$\frac{1}{\lim_{n\to\infty} c^{\frac{1}{n}}} = \lim_{n\to\infty} \frac{1}{c^{\frac{1}{n}}} = 1.$$

   This completes the proof. ■

4. Consider the sequence given by the recursion $a_{n+1} = \frac{1}{2}(a_n + a_n^{-1})$, with some initial condition $a_1 \in (-\infty, 0) \cup (0, \infty)$. Find and prove the limit (if it exists) for initial conditions $a_1 = 3$, $a_1 = 0.1$.

   Proof: Consider a value of n for which $a_n \geq 1$. For this value,

$$a_{n+1} = \frac{1}{2}(a_n + a_n^{-1}) \leq \frac{1}{2}(a_n + 1).$$

   On the other hand, consider the function $f(x) = \frac{1}{2}(x + x^{-1})$ with domain $x \in (0, \infty)$. We recognize (e.g. from completing the square) that, on this domain, the function is minimized at $x = 1$. In particular, $f(x) \geq 1$ for all $x \in (0, \infty)$. Thus,

$$a_{n+1} = \frac{1}{2}(a_n + a_n^{-1}) \geq 1.$$

   Putting together the two displayed equations, for $a_n \geq 1$ we have

$$1 \leq a_{n+1} \leq \frac{1}{2}(a_n + 1).$$

   We note that, by this bound, $a_n \geq 1$ for all $n \geq 2$ for *any* value of $a_1 \in (0, \infty)$. Iterating the upper and lower bounds, we have

$$1 \leq a_{n+1} \leq \frac{1}{2}(a_n + 1) \leq \frac{1}{2}\left(\frac{1}{2}(a_{n-1} + 1) + 1\right) = \frac{1}{4}a_{n-1} + \frac{3}{4}.$$

   Continuing to iterate, we find

$$1 \leq a_{n+1} \leq 2^{-n+1}a_2 + (1 - 2^{-n+1}).$$

   Applying the sandwich theorem, we calculate

$$1 \leq \lim_{n\to\infty} a_{n+1} \leq \lim_{n\to\infty} (2^{-n+1}a_2 + (1 - 2^{-n+1})) = 1.$$

   This completes the proof. ■

3 Subsequences

Let $\{a_n\}$ be a sequence with no convergent subsequences. Show that $|a_n| \to \infty$.

Proof: We prove this by contradiction. Assume that $|a_n|$ does *not* diverge to infinity. Then, by the definition of diverging to infinity, there exists some $M < \infty$ such that $|a_n| < M$ infinitely often. Let $1 \leq m_1 \leq m_2 \leq m_3 \leq \ldots$ be the indices satisfying $|a_{m_n}| < M$.

Set $b_n = a_{m_n}$. Then $\{b_n\}$ is a bounded sequence and so has a convergent subsequence $\{b_{k_n}\}_n$ by Bolzano-Weierstrass. But $\{a_{m_{k_n}}\}_n = \{b_{k_n}\}_n$ is in fact a convergent subsequence of $\{a_n\}$, contradicting the information given in the question. We conclude that our assumption was false, and so $|a_n|$ diverges to infinity. ■

4 Limit Superior and Limit Inferior

We define the **limit inferior** and the **limit superior** of a sequence as follows:

$$\liminf_{n\to\infty} a_n = \lim_{n\to\infty} \inf\{a_k \mid k \geq n\}$$
$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} \sup\{a_k \mid k \geq n\}.$$

1. Let $\{a_n\}$ be a bounded sequence. Show that $\liminf_{n\to\infty} a_n$ and $\limsup_{n\to\infty} a_n$ exist and are in $\mathbb{R}$.

   Proof: Define the sequence of sets $B_n = \{a_k \mid k \geq n\}$ and the sequence of numbers $b_n = \sup(B_n)$, so that

   $$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} b_n.$$

   We note that $B_1 \supset B_2 \supset \ldots$, which implies $\sup(B_1) \leq \sup(B_2) \leq \ldots$, which means that $\{b_n\}$ is monotone decreasing. Furthermore, since $\{a_n\}$ is bounded, there exists some $-\infty < M < \infty$ so that $a_n \geq M$ for all $n \in \mathbb{N}$. But this M is a lower bound for $\{a_n\}$, which means it must be a lower bound for B_n for all $n \in \mathbb{N}$, which means $b_n = \sup(B_n) \geq M$ for all $n \in \mathbb{N}$ as well.

   Thus, we have shown that $\{b_n\}$ is a monotone decreasing sequence that is bounded from below. Hence, by the monotone convergence theorem, it has a limit and so

   $$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} b_n$$

   exists.[1] ■

2. Let $\{a_n\}$ be an unbounded sequence. Show that either $\liminf_{n\to\infty} a_n = -\infty$ or $\limsup_{n\to\infty} a_n = \infty$ (or possibly both).

   Proof: Since $\{a_n\}$ is unbounded, for all $0 < M < \infty$ there exists $n = n(M)$ satisfying $|a_n| > M$. Define the subsequence $\{b_k\}$ by setting $b_k = a_{n(k)}$, so that $|b_k| > k$ for all $k \in \mathbb{N}$. Since this is an infinite sequence, we have by the pigeonhole principle that at least one of the two sets $I_+ = \{k \in \mathbb{N} \mid b_k \geq 0\}$, $I_- = \{k \in \mathbb{N} \mid b_k \leq 0\}$ is infinite.

   In the case that I_+ is infinite, write the elements $i_1 < i_2 < i_3 < \ldots$ in order and define the subsequence $\{c_\ell\}$ of $\{b_n\}$ by the formula $c_\ell = b_{i_\ell} = a_{n(i_\ell)}$. But then for all n, we have

   $$\sup\{a_k \mid k \geq n\} \geq \sup\{a_{n(i_\ell)} \mid \ell \geq n\} = \sup\{c_k \mid k \geq n\} \geq \sup\{k \mid k \geq n\} = \infty.$$

   Thus,

   $$\limsup_{n\to\infty} a_n = \infty.$$

   The case that I_- is infinite is essentially the same, with the conclusion

   $$\liminf_{n\to\infty} a_n = -\infty.$$

   This completes the proof.[2] ■

[1] The proof for the lim inf is the same. The details are at the end of this document for reference. You could also use (and prove) the identity

$$\liminf_{n\to\infty} a_n = -\limsup_{n\to\infty}(-a_n).$$

[2] As an aside, if I_-, I_+ are *both* infinite, then we have

$$\limsup_{n\to\infty} a_n = \infty, \quad \liminf_{n\to\infty} a_n = -\infty,$$

which you can check holds for sequences such as $a_n = (-n)^n$, for instance.

3. Let $\{a_n\}$, $\{b_n\}$ be two sequences. Show that

$$\liminf_{n\to\infty} a_n + \limsup_{n\to\infty} b_n \leq \limsup_{n\to\infty}(a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n.$$

Furthermore, find a pair of sequences for which the second inequality is strict.

Proof: Fix $\epsilon > 0$. Then there exists some $N = N(\epsilon)$ so that, for all $m > N$, the following inequalities all hold:

$$\begin{aligned} \frac{\epsilon}{2} + \limsup_{n\to\infty} a_n \geq a_m \geq -\frac{\epsilon}{2} + \liminf_{n\to\infty} a_n \\ \frac{\epsilon}{2} + \limsup_{n\to\infty} b_n \geq b_m \geq -\frac{\epsilon}{2} + \liminf_{n\to\infty} b_n. \end{aligned} \tag{4.1}$$

Adding the left-hand sided inequalities, we get:

$$a_m + b_m \leq \epsilon + \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n.$$

We conclude with our first desired inequality,

$$\limsup_{n\to\infty}(a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n.$$

To obtain the reverse inequality, again fix $\epsilon > 0$. Then there exists a sequence $\{k_n\}$ so that

$$b_{k_m} \geq -\frac{\epsilon}{2} + \limsup_{n\to\infty} b_n \quad \text{for all } m.$$

Chopping off the finitely-many terms in the sequence occurring before $N = N(\epsilon)$ and applying (4.1), we have for all m:

$$a_{k_m} + b_{k_m} \geq -\frac{\epsilon}{2} + \liminf_{n\to\infty} a_n - \frac{\epsilon}{2} + \limsup_{n\to\infty} b_n.$$

We conclude with our first desired inequality,

$$\limsup_{n\to\infty}(a_n + b_n) \geq \liminf_{n\to\infty} a_n + \limsup_{n\to\infty} b_n.$$

For the second inequality example, consider the sequences $a_n = (-1)^n, b_n = (-1)^{n+1}$. It is clear that $a_n + b_n = 0$ for all n, so $\limsup_{n\to\infty}(a_n + b_n) = 0$. However, $\limsup_{n\to\infty} a_n = \limsup_{n\to\infty} b_n = 1$. ∎

Details for the Analogous Limit Inferior Proof

Define the sequence of sets $B_n = \{a_k \mid k \geq n\}$ and the sequence of numbers $b_n = \inf(B_n)$, so that

$$\liminf_{n\to\infty} a_n = \lim_{n\to\infty} b_n.$$

We note that $B_1 \supset B_2 \supset \ldots$, which implies $\inf(B_1) \geq \sup(B_2) \geq \ldots$, which means that $\{b_n\}$ is monotone increasing. Furthermore, since $\{a_n\}$ is bounded, there exists some $-\infty < M < \infty$ so that $a_n \leq M$ for all $n \in \mathbb{N}$. But this M is an upper bound for $\{a_n\}$, which means it must be an upper bound for B_n for all $n \in \mathbb{N}$, which means $b_n = \inf(B_n) \leq M$ for all $n \in \mathbb{N}$ as well.

Thus, we have shown that $\{b_n\}$ is a monotone bounded sequence. Hence, by the monotone convergence theorem, it has a limit and so

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} b_n$$

exists. ∎