MAT 2125 – Homework 3

(due at midnight on March 01, in Brightspace)

1 Norms

1. Suppose $\|\cdot\|$ and $\|\cdot\|'$ are two arbitrary norms on $\mathbf{R}^d$. Prove that $\|\cdot\|''$, defined by

$$\|\mathbf{x}\|'' = \|\mathbf{x}\| + \|\mathbf{x}\|', \quad \mathbf{x} \in \mathbb{R}^d,$$

   is also a norm on $\mathbb{R}^d$.

2. A function $\|\cdot\|$ is called a *seminorm* if it satisfies all of the norm properties, *except possibly* N1 (norms are also seminorms). Let $\|\cdot\|_1, \ldots, \|\cdot\|_k$ be a collection of seminorms on $\mathbb{R}^d$. Assume that, for all $\mathbf{x} \in \mathbb{R}^d \setminus \{0\}$, there exists $i \in \{1, 2, \ldots, k\}$ so that $\|\mathbf{x}\|_i > 0$. Show that $\|\cdot\| = \displaystyle\sum_{i=1}^{k} \|\cdot\|_i$ is a norm.

3. Suppose $\{\mathbf{x}_n\}_{n=1}^{\infty}$ is a sequence in $\mathbb{R}^d$ converging to $\mathbf{y}$. Prove that $\displaystyle\lim_{n\to\infty} \|\mathbf{x}_n\| = \|\mathbf{y}\|$.

2 Closed and Open Sets

1. Fix an n by n matrix M. Show that, for any $C \geq 0$, $S = \{x \in \mathbb{R}^n : x^t M x \leq C\}$ is a closed set.

2. Consider a sequence $\{a_n\}$ with two distinct accumulation points. Show that the sequence does not have a limit.

3. Show that there exists a sequence $\{a_n\}$ whose set of accumulation points is exactly the interval $[0, 1]$. You don't need to write down an explicit formula for the sequence - you just need show that such a sequence exists.

4. Prove or disprove: there exists a sequence $\{a_n\}$ whose set of accumulation points is exactly the interval $(0, 1)$.

5. Prove that for all $\varepsilon > 0$ there exists a collection of open sets $(a_1, b_1), (a_2, b_2), \ldots$ satisfying the following two properties:

$$\bigcup_{n\in\mathbb{N}} (a_n, b_n) \supseteq \mathbb{Q} \quad \text{and} \quad \sum_{n\in\mathbb{N}} (b_n - a_n) < \varepsilon.$$

   Hint: Recall that $\sum_{n\in\mathbb{N}} 2^{-n} = 1$.

3 Compact Sets

1. Show that a finite union of compact sets is compact.

2. Show that an arbitrary intersection of compact sets is compact.