MAT 2125 – Homework 4

(due at midnight on March 26, in Brightspace)

1 Continuous Functions

1. Define $g : \mathbb{R} \to \mathbb{R}$ by
$$g(x) = \begin{cases} \frac{(-1)^n}{n} & \text{if } x = 1/n \text{ for some } n \in \mathbb{N}, \\ 0 & \text{if } x = 0. \end{cases}$$
Prove that g is continuous at 0.

2. Assume that the temperature distribution on the Earth's equator is continuous. Show that there are, at any time, antipodal points on the Earth's equator with the same temperature.

3. Suppose $f : \mathbb{R}^d \to \mathbb{R}^m$. The *pre-image* of a subset $B \subseteq \mathbb{R}^m$ under f is
$$f^{-1}(B) = \{\mathbf{a} \in A : f(\mathbf{a}) \in B\}.$$
Prove that f is continuous if and only if the pre-image of every open subset of $\mathbb{R}^m$ is an open subset of $\mathbb{R}^d$. (It is also true if "open" is replaced by "closed", but we will not ask you to prove this.)

   Hint: what is the definition of continuity for functions $f : \mathbb{R}^d \to \mathbb{R}^m$?

4. A function $f : A \to \mathbb{R}$ is said to be *Lipschitz* if there is a positive number M such that
$$|f(x) - f(y)| \leq M|x - y| \quad \forall\, x, y \in A.$$
Show that a Lipschitz function must be uniformly continuous, but that uniformly continuous functions do not have to be Lipschitz.

   Hint: for the second statement, consider the function $g : [0, 1] \to \mathbb{R}, \quad g(x) = \sqrt{x}$.

2 Differentiation

1. Let $a \in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ be defined by
$$f(x) = \begin{cases} x^2 & \text{if } x \geq 0, \\ ax & \text{if } x < 0. \end{cases}$$
For which values of a is f differentiable at $x = 0$? For which values of a is f continuous at $x = 0$?

2. Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Show that f is Lipschitz if and only if f' is bounded on (a, b).

   Hint: Apply the Mean Value Theorem to f on $[x, y] \subseteq [a, b]$ to show one of the directions.

3. If $x > 0$, show $1 + \frac{1}{2}x - \frac{1}{8}x^2 \leq \sqrt{1 + x} \leq 1 + \frac{1}{2}x$.

3 Riemann Integral

1. Using the definition of Riemann-integrability, show that $h : [a, b] \to \mathbb{R}$ defined by $h(x) = 2x + 1$ is Riemann-integrable on $[a, b]$, $b > a \geq 0$, and that that the Riemann integral of h on $[a, b]$ is $b^2 - a^2 + b - a$.

   Warning: you cannot use the rules of integration from calculus.

2. Prove *Riemann's Criterion* for a bounded function $f : [a, b] \to \mathbb{R}$, namely: f is Riemann-integrable over $[a, b]$ if and only if $\forall \varepsilon > 0$, $\exists P_\varepsilon$ a partition of $[a, b]$ such that the lower sum $L(P_\varepsilon; f)$ and the upper sum $U(P_\varepsilon; f)$ of f corresponding to P_ε satisfy $U(P_\varepsilon; f) - L(P_\varepsilon; f) < \varepsilon$.