MAT 2125 – Homework 5

(due at midnight on April 14, in Brightspace)

1 Properties of the Riemann Integral

1. Let $f : [a, b] \to \mathbb{R}$ be continuous, $f \geq 0$ on $[a, b]$, and $\int_a^b f = 0$. Show that $f(x) = 0$ for all $x \in [a, b]$.
2. Let $f : [a, b] \to \mathbb{R}$ be continuous and let $\int_a^b f = 0$. Show $\exists c \in [a, b]$ such that $f(c) = 0$.

2 Fundamental Theorem of Calculus

1. Let $f : [0, 3] \to \mathbb{R}$ be defined by
$$f(x) = \begin{cases} x & x \in [0, 1) \\ 1 & x \in [1, 2) \\ x & x \in [2, 3] \end{cases}.$$
Find $F : [0, 3] \to \mathbb{R}$, where
$$F(x) = \int_0^x f.$$
Where is F differentiable? What is F' there?
2. Compute $\dfrac{d}{dx} \displaystyle\int_{-x}^{x} e^{t^2}\, dt$.

3 Improper Integrals

1. Let $f : [a, b] \to \mathbb{R}$ be Riemann-integrable on $[a + \delta, b]$ and unbounded in the interval $(a, a + \delta)$ for every $0 < \delta < b - a$. Define
$$\int_a^b f = \lim_{\delta \to 0^+} \int_{a+\delta}^b f,$$
where $\delta \to 0^+$ means that $\delta \to 0$ and $\delta > 0$. A similar construction allows us to define
$$\int_a^b g = \lim_{\delta \to 0^+} \int_a^{b-\delta} g.$$
Such integrals are said to be *improper*; when the limits exist, they are further said to be *convergent*.

   How can the expression
$$\int_0^1 \frac{1}{\sqrt{|x|}}\, dx$$
be interpreted as an improper integral? Is it convergent? If so, what is its value?
2. For which values of s does the integral $\int_0^1 x^s\, dx$ converge? You may use the antiderivatives rules of calculus.

4 Sequences of Functions

1. Show that $\displaystyle\lim_{n\to\infty} \int_{\pi/2}^{\pi} \frac{\sin(nx)}{nx}\, dx = 0$. You may assume that sin is continuous and that $|\sin x| \leq 1$, $\forall x \in \mathbb{R}$.
2. Show that if $f_n \rightrightarrows f$ on $[a, b]$, and each f_n is continuous, then the sequence of functions
$$F_n(x) = \int_a^x f_n(t)\, dt$$
also converges uniformly on $[a, b]$.

5 Series and Power Series

1. If the series $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ both converge, does $\sum_{k=1}^{\infty} a_k b_k$ converge?

2. Find the radius of convergence for each of the following series.

   (a) $\sum_{k=0}^{\infty} (-1)^k x^{2k}$.

   (b) $\sum_{k=0}^{\infty} k x^k$.

   (c) $\sum_{k=0}^{\infty} k! x^k$.