MAT 2125
Elementary Real Analysis

Exercises – Solutions – Q21-Q26

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21. Show that

(a) $\lim_{n\to\infty}\left(\frac{1}{\sqrt{n+7}}\right)=0$;

(b) $\lim_{n\to\infty}\left(\frac{2n}{n+2}\right)=2$;

(c) $\lim_{n\to\infty}\left(\frac{\sqrt{n}}{n+1}\right)=0$, and

(d) $\lim_{n\to\infty}\left(\frac{(-1)^n n}{n^2+1}\right)=0$.

Proof.

(a) Let $\varepsilon > 0$. By the Archimedean property, there is a positive integer $N_\varepsilon > \frac{1}{\varepsilon^2}$. Then

$$\left|\frac{1}{\sqrt{n+7}} - 0\right| = \frac{1}{\sqrt{n+7}} < \frac{1}{\sqrt{n}} < \frac{1}{\sqrt{N_\varepsilon}} < \varepsilon,$$

whenever $n > N_\varepsilon$.

(b) Let $\varepsilon > 0$. By the Archimedean property, there is a positive integer $N_\varepsilon > \frac{4}{\varepsilon}$. Then

$$\left|\frac{2n}{n+2} - 2\right| = \left|-\frac{4}{n+2}\right| = \frac{4}{n+2} < \frac{4}{n} < \frac{4}{N_\varepsilon} < \varepsilon,$$

whenever $n > N_\varepsilon$.

(c) Let $\varepsilon > 0$. By the Archimedean property, there is a positive integer $N_\varepsilon > \frac{1}{\varepsilon^2}$. Then

$$\left|\frac{\sqrt{n}}{n+1} - 0\right| = \frac{\sqrt{n}}{n+1} < \frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}} < \frac{1}{\sqrt{N_\varepsilon}} < \varepsilon,$$

whenever $n > N_\varepsilon$.

(d) Let $\varepsilon > 0$. By the Archimedean property, there is a positive integer $N_\varepsilon > \frac{1}{\varepsilon}$. Then

$$\left|\frac{(-1)^n n}{n^2+1} - 0\right| = \frac{n}{n^2+1} < \frac{n}{n^2} = \frac{1}{n} < \frac{1}{N_\varepsilon} < \varepsilon,$$

whenever $n > N_\varepsilon$. ■

22. Show that $\lim\limits_{n\to\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)=0.$

Proof. Let $\varepsilon > 0$. By the Archimedean property, there is a positive integer $N_\varepsilon > \frac{1}{\sqrt{\varepsilon}}$.

Then

$$\left|\frac{1}{n} - \frac{1}{n+1} - 0\right| = \frac{1}{n(1+n)} < \frac{1}{n^2} < \frac{1}{N_\varepsilon^2} < \varepsilon,$$

whenever $n > N_\varepsilon$. ■

23. Find the limit of the following sequences:

(a) $\lim\limits_{n\to\infty}\left(\left(2+\dfrac{1}{n}\right)^2\right)$;

(b) $\lim\limits_{n\to\infty}\left(\dfrac{(-1)^n}{n+2}\right)$;

(c) $\lim\limits_{n\to\infty}\left(\dfrac{\sqrt{n}-1}{\sqrt{n}+1}\right)$, and

(d) $\lim\limits_{n\to\infty}\left(\dfrac{n+1}{n\sqrt{n}}\right)$.

Solution. We can only use the definition if we have a candidate. Throughout, we will assume that it is known that $\frac{1}{n} \to 0$.

(a) Note that $(2+\frac{1}{n})^2 = 4 + \frac{2}{n} + \frac{1}{n^2}$. Then, by theorem 14 (operations on sequences and limits),

$$\frac{2}{n} = 2 \cdot \frac{1}{n} \to 2 \cdot 0 = 0 \quad \text{and} \frac{1}{n^2} = \frac{1}{n} \cdot \frac{1}{n} \to 0 \cdot 0 = 0,$$

so that $4 + \frac{2}{n} + \frac{1}{n^2} \to 4 + 0 + 0 = 4$.

(b) Clearly,

$$\frac{-1}{n+2} \leq \frac{(-1)^n}{n+2} \leq \frac{1}{n+2}, \quad \forall n \in \mathbb{N}.$$

Note that $n+2 \geq n$ for all n so that

$$0 \leq \frac{1}{n+2} \leq \frac{1}{n}, \quad \forall n \in \mathbb{N};$$

as a result, $\frac{1}{n+2} \to 0$ by the squeeze theorem. Then $-\frac{1}{n+2} \to -0 = 0$ by theorem 14, so that $\frac{(-1)^n}{n+2} \to 0$ by the squeeze theorem.

(c) Re-write $\frac{\sqrt{n}-1}{\sqrt{n}+1} = 1 - \frac{2}{\sqrt{n}+1}$. Note that

$$0 \leq \frac{1}{\sqrt{n}+1} < \frac{1}{\sqrt{n}}, \quad \forall n \in \mathbb{N}.$$

We have seen that $\frac{1}{\sqrt{n}} \to 0$; as a result of the squeeze theorem, $\frac{1}{\sqrt{n}+1} \to 0$. Then $1 - \frac{2}{\sqrt{n}+1} \to 1 - 2 \cdot 0 = 1$, by theorem 14.

(d) Note that $n \leq n\sqrt{n} \leq n^2$ for all $n \in \mathbb{N}$ so

$$\frac{1}{n^2} \leq \frac{1}{n\sqrt{n}} \leq \frac{1}{n}, \quad \forall n \in \mathbb{N}.$$

But $\frac{1}{n}, \frac{1}{n^2}, \frac{1}{\sqrt{n}} \to 0$ (see previous problems) so that $\frac{1}{n\sqrt{n}} \to 0$ by the squeeze theorem. Furthermore,

$$\frac{n+1}{n\sqrt{n}} = \frac{1}{\sqrt{n}} + \frac{1}{n\sqrt{n}} \to 0 + 0 = 0,$$

by theorem 14. ■

24. Let $y_n = \sqrt{n+1} - \sqrt{n}$. Show that (y_n) and $(\sqrt{n}y_n)$ converge.

Proof. As

$$0 \leq \sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n+1} + \sqrt{n}} \leq \frac{1}{\sqrt{n}}, \quad \forall n \in \mathbb{N},$$

and $\frac{1}{\sqrt{n}} \to 0$, then $\sqrt{n+1} - \sqrt{n} \to 0$ by the squeeze theorem.

Note that $\sqrt{n} y_n = \sqrt{n(n+1)} - n = \frac{1}{\sqrt{1+\frac{1}{n}}+1}$ for all $n \in \mathbb{N}$. Then, according to theorem 14,

$$\lim_{n\to\infty} \sqrt{n} y_n = \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n}}+1} = \frac{1}{\lim\limits_{n\to\infty}\left(\sqrt{1+\dfrac{1}{n}}+1\right)} = \frac{1}{2},$$

since $\sqrt{1+\frac{1}{n}}+1 > 2$ for all $n \in \mathbb{N}$. ■

25. Find the limit of the following sequences:

(a) $\lim_{n\to\infty} \frac{\sin(n^2+212)}{n}$;

(b) $\lim_{n\to\infty} \frac{2n-1}{n+7}$;

(c) $\lim_{n\to\infty} q^n$, if $|q|<1$;

(d) $\lim_{n\to\infty} \sqrt[n]{n}$;

(e) $\lim_{n\to\infty} \frac{n!}{n^n}$, and

(f) $\lim_{n\to\infty} \sqrt[n]{3^n+5^n}$.

Solution.

(a) We cannot use Theorem 14 since neither limits exist. This doesn't necessarily mean that the limit of the quotient does not exist. In order to determine if it does, we need to use another approach.

By definition of the $\sin$ function (which we will define when we talk about power series), we have $-1 \leq \sin x \leq 1,\ \forall x \in \mathbb{R}$. Thus

$$-1 \leq \sin(n^2 + 212) \leq 1,\ \forall n \implies -\frac{1}{n} \leq \frac{\sin(n^2 + 212)}{n} \leq \frac{1}{n},\ \forall n.$$

As $\pm\frac{1}{n} \to 0$, we can use the Squeeze Theorem to conclude that

$$\lim_{n\to\infty} \frac{\sin(n^2 + 212)}{n} = 0.$$

(b) We cannot apply Theorem 14 directly since neither the numerator nor the denominator limits exist.

However,

$$\frac{2n-1}{n+7} = \frac{1/n \cdot (2n-1)}{1/n \cdot (n+7)} = \frac{2-1/n}{1+7/n} \quad \text{when } n \neq 0.$$

Because each of the constituent parts converge (and because the denominator is never equal to 0, either in the limit or in the sequence), repeated applications of Theorem 14 yield

$$\lim_{n\to\infty} \frac{2n-1}{n+7} = \frac{\lim_{n\to\infty}(2-1/n)}{\lim_{n\to\infty}(1+7/n)} = \frac{2-\lim_{n\to\infty} 1/n}{1+7\cdot \lim_{n\to\infty} 1/n} = \frac{2-0}{1+7\cdot 0} = 2.$$

(c) If $q = 0$, then $q^n = 0 \to 0$.

If $q \neq 0$, then $\frac{1}{|q|} > 1$. Thus, $\exists t > 0$ such that $\frac{1}{|q|} = 1 + t$.

From Bernoulli's Inequality, we have

$$\left(\frac{1}{|q|}\right)^n = (1+t)^n \geq 1 + nt, \ \forall n \in \mathbb{N},$$

so that $0 \leq |q^n| \leq |q|^n \leq \frac{1}{1+nt}$.

But $\frac{1}{1+nt} = 0$ when $n \to \infty$ (does this need to be proven?); thus $|q^n| \to 0$ by the squeeze theorem.

Since $-|q^n| \leq q^n \leq |q^n| \ \forall n$, $q^n \to 0$ by the squeeze theorem.

(d) Let $\varepsilon > 0$. Then $1+\varepsilon > 1$ and $0 < \frac{1}{1+\varepsilon} < 1$.

Claim: $n\left(\frac{1}{1+\varepsilon}\right)^n \to 0$ when $n \to \infty$. [1]

Hence, $\exists M_1 \in \mathbb{N}$ such that

$$\left|\frac{n}{(1+\varepsilon)^n} - 0\right| < 1 \text{ when } n > M_1 \implies 1 \leq n < (1+\varepsilon)^n \text{ when } n > M_1.$$

Set $N_\varepsilon = M_1$. Then $1-\varepsilon < 1 \leq n^{1/n} < 1+\varepsilon$ when $n > N_\varepsilon$. But this is precisely the same as $|n^{1/n} - 1| < \varepsilon$ when $n > N_\varepsilon$; thus $n^{1/n} \to 1$.

[1] The proof that $nq^n \to 0$ with $n \to \infty$ when $|q| < 1$ is left as an exercise; it is similar to the proof of part (c), but uses an extension of Bernoulli's Inequality which can be proven by induction:

$$(1+t)^n \geq 1 + nt + \frac{n(n-1)}{2}t^2, \text{ for } t > 0, n \geq 1.$$

(e) Since
$$0 \leq \frac{n!}{n^n} = \frac{n \cdot (n-1) \cdot \cdots \cdot 2 \cdot 1}{n \cdot n \cdot \cdots \cdot n \cdot n} \leq \frac{1}{n}, \ \forall n \in \mathbb{N},$$
and $\frac{1}{n} \to 0$, the squeeze theorem implies $\frac{n!}{n^n} \to 0$.

(f) Since
$$5^n \leq 3^n + 5^n \leq 5^n + 5^n = 2 \cdot 5^n \leq n \cdot 5^n, \ \forall n \geq 2,$$
then
$$5 \leq \sqrt[n]{3^n + 5^n} \leq \sqrt[n]{n} \cdot 5, \ \forall n \geq 2.$$
By part (d) of this question $\sqrt[n]{n} \to 1$. The squeeze theorem can then be applied to the above chain of inequalities to conclude $\sqrt[n]{3^n + 5^n} \to 5$. ■

26. Let (x_n) be a sequence of positive real numbers such that

$$\lim_{n\to\infty} x_n^{1/n} = L < 1.$$

Show $\exists r \in (0,1)$ such that $0 < x_n < r^n$ for all sufficiently large $n \in \mathbb{N}$.

Use this result to show that

$$\lim_{n\to\infty} x_n = 0.$$

Proof. Since $L < 1$, $\exists \varepsilon_0 > 0$ such that $L < L + \varepsilon_0 < 1$. Then, $\exists N_0 \in \mathbb{N}$ such that

$$|x_n^{1/n} - L| < \varepsilon_0 \quad \text{whenever } n > N_0.$$

Hence $L - \varepsilon_0 < x_n^{1/n} < L + \varepsilon_0$ for all $n > N_0$. Set $r = L + \varepsilon_0$. Then $r \in (0, 1)$ and

$$0 < x_n < r^n, \quad \forall n > N_0.$$

Let $\varepsilon > 0$. $r^n \to 0$ (do you know how to show this?), $\exists N_\varepsilon \geq N_0$ such that $r^n < \varepsilon$ whenever $n > N_\varepsilon$, hence

$$|x_n - 0| = x_n < r^n < \varepsilon$$

whenever $n > N_\varepsilon$. ■