MAT 2125
Elementary Real Analysis

Exercises – Solutions – Q27-Q30

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27. Give an example of a convergent (resp. divergent) sequence (x_n) of positive real numbers with $x_n^{1/n} \to 1$.

Proof. The sequences $(x_n) = \frac{1}{n}$ and $(x_n) = (n)$ do the trick. You should fill in the details or ask for hints if you're not sure how to show this – there are other solutions. ■

28. Let $x_1 = 1$, $x_{n+1} = \sqrt{2 + x_n}$ for $n \in \mathbb{N}$. Show that (x_n) converges and find the limit.

Proof. We show (x_n) is increasing and bounded by induction; by a theorem seen in class, (x_n) converges.

A quick computation shows $x_2 = \sqrt{3}$.

Initial case: Clearly, $1 \leq x_1 \leq x_2 \leq 2$.

Induction hypothesis: Suppose $1 \leq x_k \leq x_{k+1} \leq 2$. Then

$$3 \leq x_k + 2 \leq x_{k+1} + 2 \leq 4$$

and so

$$1 \leq \sqrt{3} \leq \sqrt{x_k + 2} \leq \sqrt{x_{k+1} + 2} \leq \sqrt{4} = 2,$$

i.e. $1 \leq x_{k+1} \leq x_{k+2} = 2$.

Hence (x_n) is increasing and bounded above by 2; as such $x_n \to x$ for some $x \in \mathbb{R}$. exists.

But

$$x = \lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \sqrt{2+x_n} = \sqrt{2+\lim_{n\to\infty} x_n} = \sqrt{2+x},$$

that is, $x^2 = 2+x$. The only solutions are $x = 2$ or $x = -1$, but $x = -1$ must be rejected since $1 \leq x_n$ for all n.

Thus, $x_n \to 2$. $\blacksquare$

29. Let $x_n = \displaystyle\sum_{k=1}^{n} \frac{1}{k^2}$ for all $n \in \mathbb{N}$.

Show that (x_n) is increasing and bounded above.

Proof. As $\frac{1}{(n+1)^2} > 0$ for all $n \in \mathbb{N}$, we have

$$x_n = \frac{1}{1^2} + \cdots + \frac{1}{n^2} \leq \frac{1}{1^2} + \cdots + \frac{1}{n^2} + \frac{1}{(n+1)^2} = x_{n+1}.$$

Furthermore, for any $k \geq 2 \in \mathbb{N}$, we have $\frac{1}{k^2} < \frac{1}{k-1} - \frac{1}{k}$. Then

$$\begin{aligned} x_n &= \frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2} \\ &\leq 1 + \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right) \\ &= 1 + 1 + 0 + \cdots + 0 - \frac{1}{n} < 2 \end{aligned}$$

for all $n \in \mathbb{N}$. Hence (x_n) is increasing and bounded above by 2. ■

30. Show that $c^{1/n} \to 1$ if $0 < c < 1$.

Proof. Let $x_n = c^{1/n}$ for all $n \in \mathbb{N}$.

Since $x_{n+1} = c^{1/(n+1)} > c^{1/n} = x_n$ for all $n \in \mathbb{N}$ (as $c < 1$), then (x_n) is increasing. Furthermore, $0 < c^{1/n} < 1^{1/n} = 1$ for all $n \in \mathbb{N}$, so (x_n) is bounded above.

Hence (x_n) converges, and $x_n \to x$, for some $x \in \mathbb{R}$. As all subsequences of a convergent sequence converge to the same limit as the convergent sequence, $x_{2n} = c^{1/2n} \to x$. As such,

$$x = \lim_{n\to\infty} c^{1/2n} = \lim_{n\to\infty} \sqrt{c^{1/n}} = \lim_{n\to\infty} \sqrt{x_n} = \sqrt{\lim_{n\to\infty} x_n} = \sqrt{x},$$

and so either $x = 0$ or $x = 1$. But as x_n increases to 1, there comes a point after which all x_n are "far" from 0 (you should mathematicize this statement...), so $x_n \to 1$. ■