MAT 2125
Elementary Real Analysis

Exercises – Solutions – Q31-Q34

Winter 2021

P. Boily (uOttawa)

31. Let (x_n) be a bounded sequence.

    For each $n \in \mathbb{N}$, let $s_n = \sup\{x_k : k \geq n\}$. If $S = \inf\{s_n\}$, show that there is a subsequence of (x_n) that converges to S.

Proof. As (x_n) is bounded, $\exists M > 0$ such that $-M < x_n < M$ for all $n \in \mathbb{N}$. By definition, $s_1 \geq s_2 \geq \cdots$ and $s_n \geq x_k$ for all $n \in \mathbb{N}$, $k \geq n$.

Hence $s_n > -M$ for all n and (s_n) is bounded below and decreasing, i.e. (s_n) is convergent.

Furthermore, for each $n \in \mathbb{N}$, as $s_n = \sup\{x_k : k \geq n\}$, $\exists k_n \in \mathbb{N}$ s.t.

$$s_n - \frac{1}{n} \leq x_{k_n} < s_n$$

(otherwise s_n is not the supremum).

The sequence (x_{k_n}) might not necessarily be a subsequence of (x_n), but by deleting the terms that are out of order, the resulting sequence, which we will also denote by (x_{k_n}) is a subsequence of (x_n).

Then

$$0 \leq |x_{k_n} - s_n| \leq \frac{1}{n}, \quad \forall n \in \mathbb{N}.$$

By the squeeze theorem,

$$0 \leq \lim_{n\to\infty} |x_{k_n} - s_n| \leq 0, \quad \text{so } \lim_{n\to\infty} |x_{k_n} - s_n| = 0.$$

From a theorem seen in class, this means that

$$\lim_{n\to\infty} x_{k_n} = \lim_{n\to\infty} s_n = S,$$

where the last equality comes from the theorem on bounded increasing/decreasing sequences. ■

32. Suppose that $x_n \geq 0$ for all $n \in \mathbb{N}$ and that $((-1)^n x_n)$ converges. Show that (x_n) converges.

Proof. Let $(-1)^n x_n \to \alpha$.

Consider its subsequences

$$\left((-1)^{2n} x_{2n}\right) = (x_{2n}) \quad \text{and} \quad \left((-1)^{2n+1} x_{2n+1}\right) = (-x_{2n+1})\,.$$

Then $x_{2n} \to \alpha$ and $(-x_{2n+1}) \to \alpha$.

But $x_{2n} \geq 0\ \forall n \in \mathbb{N}$ so $\alpha \geq 0$. Similarly, $-x_{2n+1} \leq 0\ \forall n \in \mathbb{N}$ so $\alpha \leq 0$.

Since $0 \leq \alpha \leq 0$, $\alpha = 0$.

By Theorem 14 (operations on limits),

$$\lim_{n\to\infty} |(-1)^n x_n| = |0| = 0.$$

But $|(-1)^n x_n| = x_n\ \forall n$, so $x_n \to 0$. ■

33. Show that if (x_n) is unbounded, then there exists a subsequence (x_{n_k}) such that $1/x_{n_k} \to 0$.

Proof. As (x_n) is unbounded, $\exists n_1 \in \mathbb{N}$ such that $|x_{n_1}| \geq 1$.

Moreover, $\forall k \geq 2$, $\exists n_k \in \mathbb{N}$ such that $|x_{n_k}| \geq k$ and $n_{k+1} > n_k$ (otherwise the sequence would be bounded).

Let $\varepsilon > 0$. By the Archimedean property, $\exists K_\varepsilon \in \mathbb{N}$ such that $K_\varepsilon > \frac{1}{\varepsilon}$ and

$$\left|\frac{1}{x_{n_k}} - 0\right| = \frac{1}{|x_{n_k}|} \leq \frac{1}{k} < \frac{1}{K_\varepsilon} < \varepsilon$$

whenever $k > K_\varepsilon$.

Thus, $1/x_{n_k} \to 0$. ■

34. If $x_n = \frac{(-1)^n}{n}$, find the convergent subsequence in the proof of the Bolzano-Weierstrass theorem, with $I_1 = [-1, 1]$.

Solution. We must first note that (x_n) is bounded by -1 and 1, so the question makes sense.

Let $n_1 = 1$. Then $x_{n_1} = x_1 = -1$ and length$(I_1) = 2$. Set $I_1' = [-1, 0]$ and $I_1'' = [0, 1]$.

We have

$$A_1 = \{n \in \mathbb{N} \mid n > n_1 \text{ and } x_n \in I_1'\} = \{3, 5, 7, 9, 11, \ldots\}$$

and

$$B_1 = \{n \in \mathbb{N} \mid n > n_1 \text{ and } x_n \in I_1''\} = \{2, 4, 6, 8, 10, \ldots\}.$$

Since A_1 is infinite (why?), set $I_2 = I_1' = [-1, 0]$ and $n_2 = \min A_1 = 3$, so that $x_{n_2} = -1/3$. Note that $n_2 > n_1$, $I_2 \subseteq I_1$, and length$(I_2) = 1$.

Set $I_2' = [-1, -1/2]$ and $I_2'' = [-1/2, 0]$.

We have

$$A_2 = \{n \in \mathbb{N} \mid n > n_2 \text{ and } x_n \in I_2'\} = \varnothing$$

and

$$B_2 = \{n \in \mathbb{N} \mid n > n_2 \text{ and } x_n \in I_2''\} = \{5, 7, 9, 11, 13, \ldots\}.$$

Since A_2 is finite, set $I_3 = I_2'' = [-1/2, 0]$ and $n_3 = \min B_2 = 5$, so that $x_{n_3} = -1/5$.

Note that $n_3 > n_2 > n_1$, $I_3 \subseteq I_2 \subseteq I_1$, and length$(I_3) = 1/2$.

For $k \geq 3$, we set $I'_k = [-1/2^{k-2}, -1/2^{k-1}]$ and $I''_k = [-1/2^{k-1}, 0]$. Then

$$A_k = \{n \in \mathbb{N} \mid n > n_k \text{ and } x_n \in I'_k\} = \varnothing$$

and

$$B_k = \{n \in \mathbb{N} \mid n > n_k \text{ and } x_n \in I''_k\} = \{2k+1, 2k+3, 2k+5, \ldots\}.$$

A_k is finite, so set $I_{k+1} = I''_k = [-1/2^{k-1}, 0]$.

Furthermore, $n_{k+1} = \min B_k = 2k+1$ so that $x_{n_k} = \frac{-1}{2k+1}$.

Note that $n_{k+1} > n_k > \cdots > n_2 > n_1$, $I_{k+1} \subseteq I_k \subseteq \cdots \subseteq I_2 \subseteq I_1$ and $\text{length}(I_{k+1}) = 1/2^{k-2}$.

The convergent subsequence is thus is $-1, -1/3, -1/5, \ldots \to 0$. ∎