MAT 2125
Elementary Real Analysis

Exercises – Solutions – Q31-Q34

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35. Show directly that a bounded increasing sequence is Cauchy.

Proof. Let $\varepsilon > 0$.

By completeness of $\mathbb{R}$, $x^* = \sup\{x_n \mid n \in \mathbb{N}\}$ exists as $\{x_n \mid n \in \mathbb{N}\}$ is bounded and non-empty.

In particular, $\exists M_{\frac{\varepsilon}{2}} \in \mathbb{N}$ such that

$$x^* - \frac{\varepsilon}{2} < x_{M_{\frac{\varepsilon}{2}}} \leq x^*.$$

But $x^* \geq x_n > x_{M_{\frac{\varepsilon}{2}}}$ whenever $n > M_{\frac{\varepsilon}{2}}$.

Let $N_\varepsilon = M_{\frac{\varepsilon}{2}}$. Then

$$|x_m - x_n| = |x_m - x^* + x^* - x_n| \leq |x^* - x_m| + |x^* - x_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

whenever $m, n > N_\varepsilon$. ■

36. If $0 < r < 1$ and $|x_{n+1} - x_n| < r^n$ for all $n \in \mathbb{N}$, show that (x_n) is Cauchy.

Proof. Let $\varepsilon > 0$.

By the Archimedean property, $\exists N_\varepsilon > \log_r(\varepsilon(1-r)) + 1$, i.e. $r^{N_\varepsilon - 1} < \varepsilon$.

Then

$$
\begin{aligned}
|x_m - x_n| &\leq |x_m - x_{m-1}| + \cdots + |x_{n+1} - x_n| \\
&< r^{m-1} + \cdots + r^n < \frac{r^{n-1}}{1-r} < \frac{r^{N_\varepsilon - 1}}{1-r} < \varepsilon
\end{aligned}
$$

whenever $m > n > N_\varepsilon$.

(The third last inequality holds since $r^{m-1} + \cdots + r^n$ is a geometric progression.) ■

37. If $x_1 < x_2$ and $x_n = \frac{1}{2}(x_{n-1} + x_{n-2})$ for all $n \in \mathbb{N}$, show that (x_n) is convergent and compute its limit.

Proof. We start by showing that (x_n) is a Cauchy sequence. let $L = x_2 - x_1$. Then

$$|x_n - x_{n-1}| \leq \frac{L}{2^{n-2}}$$

by induction (show this!).

Let $\varepsilon > 0$. By the Archimedean Property, $\exists N_\varepsilon \in \mathbb{N}$ such that $\frac{L}{2^{N_\varepsilon - 2}} < \varepsilon$. Then

$$\begin{aligned} |x_m - x_n| &\leq |x_m - x_{m-1}| + \cdots + |x_{n+1} - x_n| \\ &\leq \frac{L}{2^{m-2}} + \cdots \frac{L}{2^{n-1}} \leq \frac{L}{2^{n-2}} < \frac{L}{2^{N_\varepsilon - 2}} < \varepsilon \end{aligned}$$

whenever $m > n > N_\varepsilon$. Hence (x_n) is a Cauchy sequence, and so it converges.

Let $x_n \to x$. We can show, by induction (do this!), that

$$x_{2n+1} = x_1 + \frac{L}{2} + \frac{L}{2^3} + \cdots + \frac{L}{2^{2n-1}}$$

for all $n \in \mathbb{N}$. In particular,

$$\begin{aligned} x = \lim_{n\to\infty} x_{2n+1} &= x_1 + \lim_{n\to\infty}\left(\frac{L}{2} + \frac{L}{2^3} + \cdots + \frac{L}{2^{2n-1}}\right) \\ &= x_1 + \frac{L}{2}\lim_{n\to\infty}\left(1 + \frac{1}{2^2} + \cdots + \frac{1}{2^{2n-2}}\right) \\ &= x_1 + \frac{L}{2}\lim_{n\to\infty}\left(\frac{1-(1/2^2)^n}{1-(1/2^2)}\right) = x_1 + \frac{2}{3}L = \frac{1}{3}(x_1 + 2x_2). \end{aligned}$$

For instance, when $x_1 = 1$ and $x_2 = 2$, $x_n \to 5/3$. ■