MAT 2125
Elementary Real Analysis

Exercises – Solutions – Q38-Q43

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38. Show that, if $\|\cdot\|_1, \|\cdot\|_2$ are norms on $\mathbb{R}^d$ and $c_1, c_2 \in (0, \infty)$, then $c_1\|\cdot\|_1 + c_2\|\cdot\|_2$ is a norm.

Proof. Write $f(\mathbf{x}) = c_1\|\mathbf{x}\|_1 + c_2\|\mathbf{x}\|_2$. Then $f : \mathbb{R}^d \to \mathbb{R}_0^+$ since $\| \cdot \|_{1,2}$ are norms and $c_{1,2} > 0$.

The function f is a norm on $\mathbb{R}^d$ if it satisfies the following properties:

(a) $f(\mathbf{0}) = 0$;
(b) $f(a\mathbf{x}) = |a| f(\mathbf{x})$ for all $a \in \mathbb{R}, \mathbf{x} \in \mathbb{R}^d$, and
(c) $f(\mathbf{x} + \mathbf{y}) \leq f(\mathbf{x}) + f(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$.

But

$$f(\mathbf{0}) = c_1\|\mathbf{0}\|_1 + c_2\|\mathbf{0}\|_2 = c_1 \cdot 0 + c_2 \cdot 0 = 0,$$

so the first condition is met.

Similarly, for $a \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^d$, we have

$$f(a\mathbf{x}) = c_1\|a\mathbf{x}\|_1 + c_2\|a\mathbf{x}\|_2 = c_1|a|\|\mathbf{x}\|_1 + c_2|a|\|\mathbf{x}\|_2 = |a|f(\mathbf{x})$$

and the second condition is met.

The triangle inequality is proven in the same manner: for $\mathbf{x},\mathbf{y}\in\mathbb{R}^d$, we have

$$\begin{aligned} f(\mathbf{x}+\mathbf{y}) &= c_1\|\mathbf{x}+\mathbf{y}\|_1 + c_2\|\mathbf{x}+\mathbf{y}\|_2 \\ &\leq c_1(\|\mathbf{x}\|_1 + \|\mathbf{y}\|_1) + c_2(\|\mathbf{x}\|_2 + \|\mathbf{y}\|_2) \\ &= c_1\|\mathbf{x}\|_1 + c_2\|\mathbf{x}\|_2 + c_1\|\mathbf{y}\|_1 + c_2\|\mathbf{y}\|_2 \\ &= f(\mathbf{x}) + f(\mathbf{y}). \end{aligned}$$

The function f is thus a norm. ∎

39. Prove that every convergent sequence in $\mathbb{R}^d$ is bounded.

Proof. Let $(\mathbf{x}_n) \subseteq \mathbb{R}^d$ converge to $\mathbf{x} \in \mathbb{R}^d$. Then for $\varepsilon = 1$, say, $\exists N \in \mathbb{N}$ such that

$$\|\mathbf{x}_n - \mathbf{x}\| < 1 \quad \text{when } n > N.$$

Thanks to the reverse triangle inequality, we also have

$$\|\mathbf{x}_n\| - \|\mathbf{x}\| \leq \|\mathbf{x}_n - \mathbf{x}\| < 1 \quad \text{when } n > N,$$

so that $\|\mathbf{x}_n\| < \|\mathbf{x}\| + 1$ when $n > N$.

Now, set $M = \max\{\|\mathbf{x}_1\|, \ldots, \|\mathbf{x}_N\|, \|\mathbf{x}\| + 1\}$. Then $\|\mathbf{x}_n\| \leq M$ for all n and so $(\mathbf{x}_n)$ is bounded. ■

Note that the proof does not depend on the specific nature of the norm function in use, only on the sub-additivity property (triangle inequality, reverse triangle inequality) of norms in general.

40. Recall that the open ball of radius $r > 0$ centered on $\mathbf{x} \in \mathbb{R}^d$ with respect to a norm $\|\cdot\|$ is defined by

$$B_r(\mathbf{x}; \|\cdot\|) = \left\{\mathbf{y} \in \mathbb{R}^d \mid \|\mathbf{x} - \mathbf{y}\| < r\right\}.$$

A set $S \subseteq \mathbb{R}^d$ is open if for all $\mathbf{x} \in S$, $\exists r > 0$ so that $B_r(\mathbf{x}; \|\cdot\|) \subseteq S$.

Show that this definition of an open set does not depend on the norm used to define the open balls.

Proof. Let $\| \cdot \|_a$ and $\| \cdot \|_b$ be two norms on $\mathbb{R}^d$.

Let $S \subseteq \mathbb{R}^d$ be open w.r.t. to $\| \cdot \|_a$, and let $\mathbf{x} \in S$. By assumption, $\exists r > 0$ such that $B_r(\mathbf{x}; \| \cdot \|_a) \subseteq S$.

By equivalence of norms, $\exists C > 0$ s.t. $C\|\mathbf{y}\|_a \leq \|\mathbf{y}\|_b$ for all $\mathbf{y} \in \mathbb{R}^d$. Set $r' = C \cdot r > 0$. If $\|\mathbf{y}\| \in B_{r'}(\mathbf{x}; \| \cdot \|_b)$, then $\|\mathbf{x} - \mathbf{y}\|_b < r'$ and

$$C\|\mathbf{x} - \mathbf{y}\|_a \leq \|\mathbf{x} - \mathbf{y}\|_b < r' \implies \|\mathbf{x} - \mathbf{y}\|_a < \frac{r'}{C} = r,$$

which means that $\mathbf{y} \in B'_r(\mathbf{x}; \| \cdot \|_a)$.

Thus $\exists r' > 0$ such that $B_{r'}(\mathbf{x}; \| \cdot \|_b) \subseteq B_r(\mathbf{x}; \| \cdot \|_a) \subseteq S$, and so S is open with respect to $\| \cdot \|_b$. Since the norms are arbitrary, this completes the proof. ■

41. Give an open cover of $(0,1)$ with no finite subcover. Also give a sequence in $(0,1)$ without any subsequence that converges to a point in $(0,1)$.

Proof. Consider the open collection $\mathcal{U} = \left\{ \left(\frac{1}{n+2}, \frac{1}{n} \right) \mid n \in \mathbb{N} \right\}$.

The set $\mathcal{U}$ is a cover of $(0,1)$. Indeed, let $x \in (0,1)$. Then $\frac{1}{x} > 1$ and, by the Archimedean property, $\exists N \in \mathbb{N}$ such that $N < \frac{1}{x} < N+2$.

Thus $x \in (\frac{1}{N+2}, \frac{1}{N}) \in \mathcal{U}$. But no finite subset of $\mathcal{U}$ can cover $(0,1)$. Indeed, let $\{n_1 < \ldots < n_m\}$ be a finite set of integers, corresponding to the finite subset

$$\mathcal{F} = \left\{ \left(\frac{1}{n_1+2}, \frac{1}{n_1} \right), \cdots, \left(\frac{1}{n_m+2}, \frac{1}{n_m} \right) \right\} \subseteq \mathcal{U}.$$

But no real number $0 < x < \frac{1}{n_m+2}$ belongs to an element of $\mathcal{F}$, and so $\mathcal{F}$ cannot be a subcover of $(0,1)$.

Thus $(0,1)$ is not a compact subset of $\mathbb{R}$ (in the metric topology).

The sequence $\left(\frac{1}{n}\right) \subseteq (0,1)$ converges to 0, and so any of its convergent subsequences must also converge to 0 (by a theorem seen in class); as such, none of its subsequences can converge to a point in $(0,1)$. ■

42. Say that a set $K \subset \mathbb{R}^d$ is disconnected if there exist open sets $A, B \neq \varnothing$ such that $K = A \cup B$, $A \cap \overline{B} = \overline{A} \cap B = \varnothing$. Otherwise, it is connected.

    Show that $K_1 = [0, 1]$ is connected while $K_2 = (0, 1) \cup (1, 2)$ is disconnected.

    Note: the closure $\overline{A}$ of a set $A \subseteq \mathbb{R}^d$ is the smallest closed set containing A, which is to say, it is the set of all limit points of A.

Proof. That the set $K_2 = (0,1) \cup (1,2)$ is disconnected is immediate – indeed, if $A = (0,1)$, $B = (1,2)$, then $\overline{A} = [0,1]$, $\overline{B} = [1,2]$, $K_2 = A \cup B$ and

$$A \cap \overline{B} = (0,1) \cap [1,2] = \varnothing = [0,1] \cap (1,2) = \overline{A} \cap B.$$

Now assume $\exists \varnothing \neq A, B \subseteq K_1$ such that $A \cap \overline{B} = \overline{A} \cap B = \varnothing$.

Since $A, B \neq \varnothing$, $\exists a \in A, b \in B$ such that $a < b$ (re-label the sets A, B if necessary). By assumption, $[a, b] \subseteq K_1$.

Consider the sets $A_0 = A \cap [a, b], B_0 = B \cap [a, b] \subseteq K_1 = [0, 1]$. Since $A_0 \neq \varnothing$ is bounded, its supremum $c = \sup A_0$ exists, with $c \in \overline{A_0} \subseteq \overline{A}$.

As A_0 is bounded above by b and below by a, we must have $a \leq c \leq b$. However, $\overline{A} \cap B = \varnothing$ by assumption, so $c \notin B$. But $b \in B$, so $c < b$.

If $c \in A$, then $\exists r > 0$ such that $B_r(c) \subseteq A \implies B_r(c) \cap [a, b] \subseteq A_0$.

Let $\varepsilon = \min\{\frac{r}{2}, \frac{b-c}{2}\} > 0$.

Then $c + \varepsilon \in A$ since $c < c + \varepsilon < c + r$.

Furthermore, $c + \varepsilon \in [a, b]$ since $a \leq c < c + \varepsilon < b$.

Thus $c + \varepsilon \in A \cap [a, b] = A_0$, which contradicts the fact that c is an upper bound of A_0. We must then have $c \notin A$.

We have thus found a real number $c \in [a, b] \subseteq K_1 = [0, 1]$ which is in neither A nor B; consequently, $A \cup B \neq K_1 = [0, 1]$.

As A and B were arbitrary, K_1 must then be connected. ■

43. We say that $K \subset \mathbb{R}^d$ is path-connected if for all $\mathbf{x}_1, \mathbf{x}_2 \in K$, there exists a continuous function $p : [0, 1] \to K$ such that $p(0) = \mathbf{x}_1$ and $p(1) = \mathbf{x}_2$.

    Let K be a compact, path-connected set and let $f : K \to \mathbb{R}$ be continuous on K.

    Show $\exists \mathbf{x}_{\text{min}}, \mathbf{x}_{\text{max}} \in K$ such that $f(K) = [f(\mathbf{x}_{\text{min}}), f(\mathbf{x}_{\text{max}})]$.

Proof. Since f is continuous and since K is compact and path-connected, $f(K)$ is both compact and path-connected.

We start by showing compactness. Let $\mathcal{U} = \{U_i \mid i \in I\}$ be an open cover of $f(K)$. Consider the collection $\mathcal{W} = \{f^{-1}(U_i) \mid i \in I\}$, where

$$f^{-1}(V) = \{\mathbf{y} \in K \mid f(\mathbf{y}) \in V\}.$$

If $\mathbf{x} \in K$, then $f(\mathbf{x}) \in f(K)$. Since $\mathcal{U}$ is an open cover of $f(K)$, $\exists U_i \in \mathcal{U}$ such that $f(\mathbf{x}) \in U_i$, which means that $\mathbf{x} \in f^{-1}(U_i)$. But $\mathbf{x}$ was arbitrary, and so $\mathcal{W}$ is a cover of K. Now, let $\mathbf{x} \in K$. Then $\exists U_i \in \mathcal{U}$ such that $f(\mathbf{x}) \in U_i$. Since U_i is open in $\mathbb{R}$, $\exists r > 0$ such that $B_r(f(\mathbf{x})) \subseteq U_i$.

We then have

$$f^{-1}\left(B_r(f(\mathbf{x}))\right) \subseteq f^{-1}(U_i).$$

But f is continuous at $\mathbf{x}$, so for $\varepsilon = r > 0$, $\exists \delta_\varepsilon = \delta_r > 0$ such that $|f(\mathbf{y}) - f(\mathbf{x})| < r$ when $\mathbf{y} \in B_{\delta_r}(\mathbf{x})$.

Thus, if $\mathbf{y} \in B_{\delta_r}(\mathbf{x})$, then $f(\mathbf{y}) \in B_r(f(\mathbf{x}))$; i.e.

$$B_{\delta_r}(\mathbf{x}) \subseteq f^{-1}\left(B_r(f(\mathbf{x}))\right) \subseteq f^{-1}(U_i),$$

which is to say that $f^{-1}(U_i)$ is open in $\mathbb{R}^d$.

Consequently, $\mathcal{W}$ is an open cover of K. But K is compact, so $\mathcal{W}$ admits a finite subcover $\{f^{-1}(U_{i_1}), \ldots, f^{-1}(U_{i_k})\}$.

The sub-collection $\mathcal{U}' = \{U_{i_1}, \ldots, U_{i_k}\}$ is a subcover of $f(K)$. Let $f(\mathbf{x}) \in f(K)$. Then $\mathbf{x} \in K$, so $\exists U_{i_m} \in \mathcal{U}'$ such that $\mathbf{x} \in f^{-1}(U_{i_m})$. But this is precisely the same as saying that $f(\mathbf{x}) \in U_{i_m}$. Thus $\mathcal{U}'$ is a finite subcover and $f(K)$ is compact.

To show that it is path-connected, consider two arbitrary points $f(\mathbf{x}_1), f(\mathbf{x}_2) \in f(K)$. Since K is path-connected, $\exists p : [0,1] \to K$ continuous with $p(0) = \mathbf{x}_1$ and $p(1) = \mathbf{x}_2$.

The composition $\varphi = f \circ p : [0,1] \to \mathbb{R}$ is continuous, being the composition of continuous functions, with

$$\varphi(0) = f(p(0)) = f(\mathbf{x}_1) \quad \text{and} \quad \varphi(1) = f(p(1)) = f(\mathbf{x}_2);$$

thus φ is a path from $\mathbf{x}_0$ to $\mathbf{x}_1$. Consequently, $f(K)$ is path-connected.

But the only subsets of $\mathbb{R}$ that are compact and path-connected are precisely the closed and bounded intervals.

If $f(K)$ is a singleton or $\varnothing$, there is nothing to show. So we assume that there are at least two distinct elements $\alpha \neq \beta \in f(K)$.

To show that $f(K)$ is bounded, let $\xi \in f(K)$ and consider the open interval

$$B_n(\xi) = \{x \in \mathbb{R} \mid |x - \xi| < n\} \subseteq \mathbb{R}.$$

The collection $\{B_i(\xi) \mid i \in \mathbb{N}\}$ is an open cover of $f(K)$.

By compactness of $f(K)$, the collection admits a finite subcover $\{B_{n_1}(\xi), \ldots, B_{n_k}(\xi)\}$. Let $M = \max\{n_1, \ldots, n_k\}$. Then

$$f(K) \subseteq \bigcup_{j=1}^{k} B_{n_j}(\xi) = B_M(\xi).$$

To show that $f(K)$ is closed, we show that its complement $\mathbb{R} \setminus f(K)$ is open. Because $f(K)$ is bounded, we know that $\exists \gamma \notin f(K)$.

For $\eta \in f(K)$, let $r = \frac{|\gamma-\eta|}{2}$ and set $V_\eta = B_r(\gamma)$ and $W_\eta = B_r(\eta)$. Note that $V_\eta \cap W_\eta = \varnothing$, by construction.

The collection $\mathcal{W} = \{W_\eta \mid \eta \in f(K)\}$ is an open cover of $f(K)$. Since $f(K)$ is compact, it admits a finite subcover $\{W_{\eta_1}, \ldots, W_{\eta_n}\}$. Consider the corresponding finite intersection

$$V = \bigcap_{j=1}^{n} V_{\eta_j}.$$

Because all the V_{η_j} are open balls centered at γ, the intersection is simply the $V_{\eta*}$ with radius

$$\eta* = \arg_{\eta_j} \min \left\{ \frac{|\eta_1 - \gamma|}{2}, \ldots, \frac{|\eta_n - \gamma|}{2} \right\}.$$

Now, if

$$z \in f(K) \subseteq W_{\eta_1} \cup \cdots \cup W_{\eta_n},$$

then $z \in W_{\eta_j}$ for some $1 \leq j \leq n$. Thus $z \notin V_{\eta_j}$ and

$$z \notin V_{\eta*} = V_{\eta_1} \cap \cdots \cap V_{\eta_n}.$$

Consequently, $V_{\eta_j} \cap f(K) = \varnothing$ and $B_{\frac{|\eta* - \gamma|}{2}}(\gamma) \subseteq \mathbb{R} \setminus f(K)$, which means that $\mathbb{R} \setminus f(K)$ is open and thus that $f(K)$ is closed.

To show that $f(K) \subseteq \mathbb{R}$ is an interval, first note that it is bounded and closed (as shown above), and so

$$\inf\{f(K)\} = \min\{f(K)\} \in f(K) \quad \text{and} \quad \sup\{f(K)\} = \max\{f(K)\} \in f(K);$$

thus $f(K) \subseteq [\alpha, \beta] = [\inf f(K), \sup f(K)]$.

Now, let $\alpha \leq z \leq \beta$.

Since $\alpha, \beta \in f(K)$ and $f(K)$ is path-connected, $\exists$ a continuous path $\psi : [0,1] \to f(K) \subseteq \mathbb{R}$ with $\psi(0) = \alpha$ and $\psi(1) = \beta$.

By the intermediate value theorem, $\exists \mu \in [0,1]$ such that $\psi(\mu) = z$, meaning that $z \in f(K)$. Thus $[\alpha, \beta] \subseteq f(K)$.

We have thus shown that $f(K) = [\alpha, \beta]$.

As $f : K \to f(K)$ is surjective (onto), $\exists \mathbf{x}_{\text{min}}, \mathbf{x}_{\text{max}} \in K$ such that $f(\mathbf{x}_{\text{min}}) = \alpha$ and $f(\mathbf{x}_{\text{max}}) = \beta$, which completes the proof. ■