MAT 2125
Elementary Real Analysis

Exercises – Solutions – Q44-Q48

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44. Let $f : \mathbb{R} \to \mathbb{R}$ and let $c \in \mathbb{R}$.

Show that $\lim_{x \to c} f(x) = L$ if and only if $\lim_{x \to 0} f(x + c) = L$.

Proof.

$$\lim_{x\to c} f(x) = L$$

$$\Updownarrow$$

$$\forall \varepsilon > 0, \exists \delta_\varepsilon > 0 \text{ s.t. } |f(x) - L| < \varepsilon \text{ when } 0 < |x - c| < \delta_\varepsilon$$

$$\Updownarrow$$

$$\text{Set } x = y + c: \ \forall \varepsilon > 0, \exists \delta_\varepsilon > 0 \text{ s.t. } |f(y + c) - L| < \varepsilon \text{ when } 0 < |y| < \delta_\varepsilon$$

$$\Updownarrow$$

$$\forall \varepsilon > 0, \exists \delta_\varepsilon > 0 \text{ s.t. } |f(y + c) - L| < \varepsilon \text{ when } 0 < |y - 0| < \delta_\varepsilon$$

$$\Updownarrow$$

$$\lim_{y\to 0} f(y + c) = L.$$

There you go. ■

45. Show $\lim_{x \to c} x^3 = c^3$ for any $c \in \mathbb{R}$.

Proof. If $|x-c|<1$, then $|x|<|c|+1$.

Let $\varepsilon>0$. Set $\delta_\varepsilon=\min\{1,\frac{\varepsilon}{3|c|^2+3|c|+1}\}$.

Then

$$\begin{aligned}
|x^3-c^3| &= |x-c||x^2+cx+c^2| \\
&\leq |x-c|\left(|x|^2+|c||x|+|c|^2\right) \\
&< |x-c|\left((|c|+1)^2+|c|(|c|+1)+|c|^2\right) \\
&= |x-c|\left(3|c|^2+3|c|+1\right) \\
&< \delta_\varepsilon\cdot\left(3|c|^2+3|c|+1\right) \leq \frac{\varepsilon}{3|c|^2+3|c|+1}\cdot\left(3|c|^2+3|c|+1\right)=\varepsilon,
\end{aligned}$$

whenever $0<|x-c|<\delta_\varepsilon$ and $x\in\mathbb{R}$. ■

46. Use either the $\varepsilon - \delta$ definition of the limit or the Sequential Criterion for limits to establish the following limits:

(a) $\displaystyle\lim_{x\to 2} \frac{1}{1-x} = -1$;

(b) $\displaystyle\lim_{x\to 1} \frac{x}{1+x} = \frac{1}{2}$;

(c) $\displaystyle\lim_{x\to 0} \frac{x^2}{|x|} = 0$, and

(d) $\displaystyle\lim_{x\to 1} \frac{x^2-x+1}{x+1} = \frac{1}{2}$

Proof.

(a) Let $\varepsilon > 0$ and set $\delta_\varepsilon = \min\{\frac{1}{2}, \frac{\varepsilon}{2}\}$. Then

$$0 < |x-2| < \delta_\varepsilon \implies |x-2| < \frac{1}{2} \iff \frac{3}{2} < x < \frac{5}{2}$$
$$\iff \frac{1}{2} < x-1 < \frac{3}{2} \iff \frac{1}{x-1} < 2.$$

Thus

$$\left| \frac{1}{1-x} - (-1) \right| = \frac{1}{|x-1|}|x-2| = \frac{1}{x-1}|x-2| < 2\delta_\varepsilon < \varepsilon$$

whenever $0 < |x-2| < \delta_\varepsilon$ and $x \in \mathbb{R}$. (Note that if $0 < |x-2| < \delta_\varepsilon$, we've seen that $x > \frac{3}{2}$ and so that $|x-1| = x-1$. This explains why we have gotten rid of the absolute values above.)

(b) Let $\varepsilon > 0$ and set $\delta_\varepsilon = \min\{\frac{1}{2}, 3\varepsilon\}$. Then

$$0 < |x-1| < \delta_\varepsilon \implies |x-1| < \frac{1}{2} \iff \frac{1}{2} < x < \frac{3}{2}$$
$$\iff 3 < 2(x+1) < 5 \iff \frac{1}{2(x+1)} < \frac{1}{3}.$$

Thus

$$\left|\frac{x}{1+x} - \frac{1}{2}\right| = \frac{1}{2|x+1|}|x-1| = \frac{1}{2(x+1)}|x-1| < \frac{1}{3}\delta_\varepsilon < \varepsilon$$

whenever $0 < |x-1| < \delta_\varepsilon$ and $x \in \mathbb{R}$. (Note that if $0 < |x-1| < \delta_\varepsilon$, we've seen that $2(x+1) > 3$ and so that $2|x+1| = 2(x+1)$. This explains why we have gotten rid of the absolute values above.)

(c) Let $(x_n) \subseteq \mathbb{R}$ be a sequence s.t. $x_n \to 0$ and $x_n \neq 0$ for all n. Then

$$\frac{x_n^2}{|x_n|} = \frac{|x_n|^2}{|x_n|} = |x_n| \to 0,$$

by theorem 14. By another theorem, the limit must be thus 0.

(d) Let $\varepsilon > 0$ and set $\delta_\varepsilon = \min\{\frac{1}{2}, \frac{3}{2}\varepsilon\}$. Then

$$0 < |x-1| < \delta_\varepsilon \implies |2x-1| < 2 \text{ and } \left|\frac{1}{2(x+1)}\right| < \frac{1}{3}.$$

Thus ,whenever $0 < |x-1| < \delta_\varepsilon$ and $x \in \mathbb{R}$, we have

$$\left|\frac{x^2-x+1}{x+1} - \frac{1}{2}\right| = \left|\frac{2x-1}{2(x+1)}\right| |x-1| < \frac{2}{3}|x-1| < \frac{2}{3}\delta_\varepsilon < \varepsilon. \quad \blacksquare$$

47. Show that the following limits do not exist:

(a) $\displaystyle\lim_{x\to 0}\frac{1}{x^2}$, with $x>0$;

(b) $\displaystyle\lim_{x\to 0}\frac{1}{\sqrt{x}}$, with $x>0$;

(c) $\displaystyle\lim_{x\to 0}(x+\operatorname{sgn}(x))$, and

(d) $\displaystyle\lim_{x\to 0}\sin(1/x^2)$, with $x>0$.

Proof. In each instance, we only give some sequence(s) for which theorem 26 shows the limit does not exist.

(a) $x_n = \frac{1}{n} \to 0$, but $f(x_n) = \frac{1}{1/n^2} = n^2 \to \infty$.

(b) $x_n = \frac{1}{n} \to 0$, but $f(x_n) = \frac{1}{1/\sqrt{n}} = \sqrt{n} \to \infty$.

(c) $x_n = \frac{1}{n}, y_n = -\frac{1}{n} \to 0$, but $f(x_n) = \frac{1}{n} + 1 \to 1, f(y_n) = -\frac{1}{n} - 1 \to -1$.

(d) $x_n = \sqrt{\frac{2}{(4n+1)\pi}}, y_n = \sqrt{\frac{2}{(4n+3)\pi}} \to 0$ but

$$f(x_n) = \sin\left(\frac{4n+1}{2}\pi\right) \to 1, \; f(y_n) = \sin\left(\frac{4n+3}{2}\pi\right) \to -1. \quad \blacksquare$$

48. Let $c \in \mathbb{R}$ and let $f : \mathbb{R} \to \mathbb{R}$ be such that $\lim_{x \to c} (f(x))^2 = L$.

Show that if $L = 0$, then $\lim_{x \to c} f(x) = 0$.

Show that if $L \neq 0$, then f may not have a limit at c.

Proof. If $\lim_{x\to c} (f(x))^2 = 0$ then $\forall \eta > 0$, $\exists \delta_\eta > 0$ such that

$$|f(x)|^2 = \left|(f(x))^2 - 0\right| < \eta$$

whenever $0 < |x - c| < \delta_\eta$. Let $\varepsilon > 0$.

By definition of the real numbers, $\exists \eta_\varepsilon > 0$ such that $\varepsilon = \sqrt{\eta_\varepsilon}$. Set $\delta_\varepsilon = \delta_{\eta_\varepsilon}$. Then

$$|f(x) - 0| = |f(x)| = \sqrt{|f(x)|^2} < \sqrt{\eta_\varepsilon} = \varepsilon$$

whenever $0 < |x - c| < \delta_\varepsilon$.

Now, consider the function $f:\mathbb{R}\to\mathbb{R}$ defined by

$$f(x)=\begin{cases}1 & \text{if } x\geq 0\\ -1 & \text{if } x<0\end{cases}.$$

Then $(f(x))^2\equiv 1$ and

$$\lim_{x\to 0}(f(x))^2=\lim_{x\to 0}1=1.$$

But $\lim_{x\to 0}f(x)$ does not exist since $(x_n)=(\frac{1}{n}), (y_n)=(-\frac{1}{n})$ are sequences such that $x_n,y_n\to 0$, $x_n,y_n\neq 0$ for all n and

$$f(x_n)=-1\to -1\neq 1\leftarrow 1=f(y_n). \quad \blacksquare$$