MAT 2377 – Assignment 1 – Solutions

Total = 100 marks

Solution 1. [30 marks] Note that this is an experiment with equally likely outcomes and the order of the selection is not important. Let A be the event that John is in Group I and B be the event that Tom is in Group I.

(a) [6 marks] To make A occur, one position in Group I must be taken by John; the other 3 positions are to be taken by other students. Thus, by the frequency method,

$$P(A) = \frac{n(A)}{n(S)} = \frac{{}_8C_3}{{}_9C_4} = \frac{56}{126} = 0.444$$

(b) [9 marks] If A has occurred, Group I only has 3 open positions. To make B occur given A, one remaining position must be taken by Tom. Thus,

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{{}_7C_2}{{}_8C_3} = 0.375$$

(c) [15 marks] The question can be expressed as

$$P\left[(A \cap B) \cup (A^c \cap B^c)\right] = P(A \cap B) + P(A^c \cap B^c) = \frac{{}_7C_2 +_7 C_4}{{}_9C_4} = 0.444$$

Solution 2. [not marked] Let F be the event of having an alcoholic father and M the event of having an alcoholic mother. We know that $P(F) = 0.42$, $P(M) = 0.08$ and $P(F \cup M) = 0.48$.

(a) The probability of having two alcoholic parents is

$$P(F \cap M) = P(F) + P(M) - P(F \cap M) = 0.42 + 0.08 - 0.48 = 0.02.$$

(b) The probability of having an alcoholic mother but not an alcoholic father is

$$P(M \cap F^c) = P(M) - P(M \cap F) = 0.08 - 0.02 = 0.06.$$

(c) The probability of having an alcoholic mother, if they have an alcoholic father is

$$P(M|F) = \frac{P(M \cap F)}{P(F)} = \frac{0.02}{0.42} = 0.0476.$$

(d) The probability of having an alcoholic mother, if they do not have an alcoholic father is

$$P(M|F^c) = \frac{P(M \cap F^c)}{P(F^c)} = \frac{0.06}{1 - 0.42} = 0.1034.$$

Solution 3. [30 marks] Let the events A, B, C be that the machine was manufactured by company A, by company B and by company C, respectively. Let D be the event that the machine is defective. We have $P(A) = 0.75$, $P(B) = 0.2$ and $P(C) = 0.05$. Furthermore, the defective rates are $P(D|A) = 0.04$, $P(D|B) = 0.05$ and $P(D|C) = 0.08$.

(a) [15 marks] Given that the machine is defective, the probability that it was made by A is

$$\begin{aligned} P(A|D) &= \frac{P(A \cap D)}{P(D)} \\ &= \frac{P(D|A)P(A)}{P(D|A)P(A) + P(D|B)P(B) + P(D|C)P(C)} \\ &= \frac{(0.04)(0.75)}{(0.04)(0.75) + (0.05)(0.2) + (0.08)(0.05)} \\ &= 0.682. \end{aligned}$$

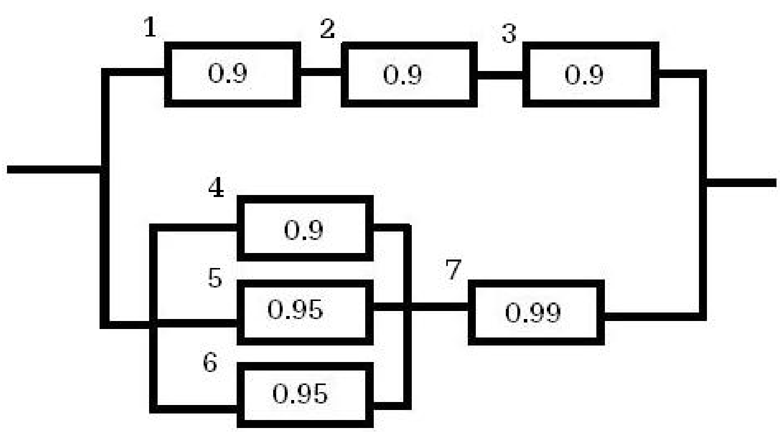
(b) [15 marks] The probability that it was made by company A and it is not defective is

$$P(A \cap D^c) = P(D^c|A)P(A) = (1 - 0.04)(0.75) = 0.72,$$

or alternatively we could have computed the probability as follows:

$$P(A \cap D^c) = P(A) - P(A \cap D) = 0.75 - (0.04)(0.75) = 0.72.$$

Solution 4. [not marked] Let us label the components as follows.



Define event E_i="component i works". We decompose the circuit into sub-circuits. Consider the components 1, 2 and 3 which are assembled into series. We denote this component as 8. Thus,

$$P(E_8) = P(E_1 \cap E_2 \cap E_3) = P(E_1)P(E_2)P(E_3) = (0.9)^3 = 0.729.$$

Consider the components 4, 5 and 6 which are assembled into parallel. We denote this component as 9. So

$$\begin{aligned} P(E_9) &= P(E_4 \cup E_5 \cup E_6) \\ &= 1 - P(E_4')P(E_5')P(E_6') \\ &= 1 - (0.1)(0.05)(0.05) = 0.99975. \end{aligned}$$

Consider components 9 and 7 which are assembled in series. We denote this component as 10. Thus,

$$P(E_{10}) = P(E_9 \cap E_7) = P(E_9)P(E_7) = (0.99975)(0.99) = 0.9897525.$$

The circuit is composed of component 10 and 8, which are assembled in parallel. Therefore, the probability that the circuit operates is

$$P(E_8 \cup E_{10}) = 1 - P(E_8')P(E_{10}') = 1 - (1 - 0.729)(1 - 0.9897525) = 0.9972.$$

Solution 5. [40 marks]

(a) [4 marks] Apparently, $f(x) \geq 0$ and $\sum_x f(x) = \frac{2}{8} + \frac{3}{8} + \frac{2}{8} + \frac{1}{8} = 1$. Thus, $f(x)$ is a pmf.

(b) [6 marks] $P(X < 1) = P(X = -1) + P(X = 0) = f(-1) + f(0) = \frac{5}{8}$
$P(X \leq 1) = P(X < 1) + P(X = 1) = \frac{5}{8} + f(1) = \frac{7}{8}$
$P(X < 0.5 \text{ or } X > 2) = P(X \leq 0) + P(X = 3) = f(-1) + f(0) + f(3) = \frac{6}{8}$

(c) [10 marks]

$$F(x) = \begin{cases} 0, & x < -1; \\ 2/8, & -1 \leq x < 0; \\ 5/8, & 0 \leq x < 1; \\ 7/8, & 1 \leq x < 3; \\ 1, & x \geq 3. \end{cases}$$

(d) [20 marks] The mean is given by

$$E(X) = -1(2/8) + 0(3/8) + 1(2/8) + 3(1/8) = 0.375$$

Note that $E(X^2) = -1^2(2/8) + 0^2(3/8) + 1^2(2/8) + 3^2(1/8) = 1.625$. Therefore,

$$\mathrm{Var}(X) = E(X^2) - [E(X)]^2 = 1.625 - 0.375^2 = 1.484$$

.