uOttawa

Department of
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This exam contains 13 pages (including this cover page and tables) and 38 questions.

Read carefully: if you submit answers for this online examination, you promise and certify that

- all work done during the assigned exam period will be done entirely by yourself, with no help from others;
- you will not communicate with anybody except the professor during the exam, for exam- and course-related questions;
- you will not consult any people, sources, writings, or Internet resources other than the course textbook, your self-made review sheets, your course notes, the course documents made available on Brightspace and/or the course website, and the exam itself;
- you will not provide information related to the exam's contents to other people until 24 hours after the exam is over.

We will not be monitoring your adherence to these instructions; we are asking, instead, for your word of honour that you will follow the directives and work on this examination alone.

True or false question are worth 2 marks each; all other questions are worth 3 marks. There are no part marks for the True of False questions. For the other questions, the marking scheme will be as discussed in class: 0, 1, 2, or 3, depending on the quality of your answer. You must provide **complete**, **clear**, and **precise** solutions to the questions to score full marks. Please provide your exam answers in the examination booklet that has been made available on Brigthspace and/or the course website.

IMPORTANT: every student has been assigned a different exam (based on the last 5 digits of your student number). You need to answer the questions of the exam that has been specifically assigned to you.

When you have answered all the questions and you are ready to upload the answers to Brightspace, save your answers to the file `MT_Answers_ALL.pdf`. We suggest that you print and scan or take a screenshot (with time stamp) of your completed answer sheet before uploading to Brightspace (in the "Assignment" section, where you upload your assignments), and that you keep a copy of these documents (with time stamp). Should Brightspace's upload functionality not work when you are ready to submit, wait a little bit and try again; as a last recourse, please email your exam booklet to `asmi28@uottawa.ca` (section A) or `pboily@uottawa.ca` (section B).

No worries. You've got this.

1. **True or False Statement:** determine the validity of the following statement. If it is false, provide a counter-example.

   If $A \subseteq \mathbb{R}$ is countable and $A \subseteq B$, then B is countable.

   Answer False; $A = \mathbb{N}$ is countable and $A \subseteq B = \mathbb{R}$, but $B = \mathbb{R}$ is uncountable.

2. **True or False Statement:** determine the validity of the following statement. If it is false, provide a counter-example.

   If $A \subseteq \mathbb{R}$ is uncountable and $A \subseteq B$, then B is uncountable.

   Answer True.

3. **True or False Statement:** determine the validity of the following statement. If it is false, provide a counter-example.

   $\mathbb{Q}$ is complete.

   Answer False; the set $\{x \in \mathbb{Q} \mid x^2 < 2\}$ is bounded but does not admit a supremum in $\mathbb{Q}$.

4. **True or False Statement:** determine the validity of the following statement. If it is false, provide a counter-example.

   Every sequence $(x_n) \subseteq \mathbb{R}$ has a convergent subsequence.

   Answer False; the sequence $(x_n) = (n)$, for instance, does not admit convergent subsequences, since all of its subsequences are unbounded.

5. **True or False Statement:** determine the validity of the following statement. If it is false, provide a counter-example.

   Every bounded sequence $(x_n) \subseteq \mathbb{R}$ has a convergent subsequence.

   Answer True (Bolzano-Weierstrass).

6. **True or False Statement:** determine the validity of the following statement. If it is false, provide a counter-example.

   Every Cauchy sequence $(x_n) \subseteq \mathbb{R}$ is convergent.

   Answer True; the Cauchy sequences in $\mathbb{R}$ are exactly those sequences that are convergent in $\mathbb{R}$.

7. **True or False Statement:** determine the validity of the following statement. If it is false, provide a counter-example.

   Every monotone sequence $(x_n) \subseteq \mathbb{R}$ is a Cauchy sequence.

   Answer False; the sequence $(x_n) = (n)$ is monotone increasing, but it is not a Cauchy sequence.

8. **True or False Statement:** determine the validity of the following statement. If it is false, provide a counter-example.

   Every convergent sequence $(x_n) \subseteq \mathbb{R}$ is bounded.

   Answer True; all convergent sequences are bounded.

9. **True or False Statement:** determine the validity of the following statement. If it is false, provide a counter-example.

   If $a_n < b_n < c_n$ for all n, and $a_n \to A, b_n \to B, c_n \to C$, then $A < B < C$.

   Answer False; $a_n = 1/n$, $b_n = 2/n$ and $c_n = 3/n$ are such that $a_n < b_n < c_n$ for all n, but $a_n, b_n, c_n \to 0$ and $0 \not< 0 \not< 0$.

10. **True or False Statement:** determine the validity of the following statement. If it is false, provide a counter-example.

    For sequences $(x_n), (y_n)$, $\liminf_{n\to\infty}(x_n + y_n) = \liminf_{n\to\infty} x_n + \liminf_{n\to\infty} y_n$.

    Answer False; consider $x_n = (-1)^n$ and $y_n = (-1)^{n+1}$ for all $n \in \mathbb{N}$ – then $x_n + y_n = 0$ for all n, and

    $$\liminf_{n\to\infty}(x_n + y_n) = 0, \quad \text{and} \liminf_{n\to\infty} x_n + \liminf_{n\to\infty} y_n = -1 - 1 = -2.$$

11. **True or False Statement:** determine the validity of the following statement. If it is false, provide a counter-example.

    If $(|x_n|)$ converges, then (x_n) converges.

    Answer False: $|(-1)^n|$ converges to 1, but $(-1)^n$ does not converge.

12. **True or False Statement:** determine the validity of the following statement. If it is false, provide a counter-example.

    If x_n doesn't converge, then x_n is unbounded.

    Answer False; $(-1)^n$ doesn't converge, but it is bounded by 2, say.

13. **True or False Statement:** determine the validity of the following statement. If it is false, provide a counter-example.

    If $(x_n) \subseteq \mathbb{R}$ is unbounded, either $\lim_{n\to\infty} x_n = \infty$ or $\lim_{n\to\infty} x_n = -\infty$.

    Answer False; $(-1)^n n$ is unbounded, but neither $\lim_{n\to\infty}(-1)^n n = \infty$ nor $\lim_{n\to\infty}(-1)^n n = -\infty$.

14. **True or False Statement:** determine the validity of the following statement. If it is false, provide a counter-example.

    If $(x_n) \subseteq \mathbb{R}^d$ and $(x_n[i])$ converges to a limit $S[i]$ for each $i \in \{1, 2, \ldots, d\}$, then (x_n) converges to $S = (S[1], \ldots, S[d])$.

    Answer True.

15. **True or False Statement:** determine the validity of the following statement. If it is false, provide a counter-example.

    If a subset of $\mathbb{R}^d$ is not open, it is closed.

    Answer False; $[0, 1)$ is not open in $\mathbb{R}$, but it is not closed in $\mathbb{R}$.

16. **True or False Statement:** determine the validity of the following statement. If it is false, provide a counter-example.

    If $A \subset \mathbb{R}^d$ is an open set, it can be written as a finite union of open balls.

    Answer False; the set $\mathbb{R}$ is open in $\mathbb{R}$, but it cannot be write as a finite union of open balls (each of which has to have a finite radius).

17. **True or False Statement:** determine the validity of the following statement. If it is false, provide a counter-example.

    Every nonempty open set $A \subset \mathbb{R}^d$ contains an open ball of radius strictly greater than 0.

    Answer True.

18. **True or False Statement:** determine the validity of the following statement. If it is false, provide a counter-example.

    Every nonempty closed set $A \subset \mathbb{R}^d$ contains a closed ball of radius strictly greater than 0.

    Answer False; $\{1\}$ is closed in $\mathbb{R}$, but it contains no closed ball of radius strictly greater than 0.

19. **True or False Statement:** determine the validity of the following statement. If it is false, provide a counter-example.

    Every boundary point of $A \subset \mathbb{R}^d$ is an accumulation (cluster) point.

    Answer False; 1 is a boundary point of $\{1\}$, but $\{1\}$ has no accumulation point.

20. **True or False Statement:** determine the validity of the following statement. If it is false, provide a counter-example.

    Every isolated point of $A \subset \mathbb{R}^d$ is a boundary point.

    Answer True.

21. **Short Answer Question:** provide a proof of the following statement.

    Let $S \neq \varnothing$ be a bounded subset of $\mathbb{R}$. Let $a > 0$. Define $aS = \{as : s \in S\}$. Prove that $\inf(aS) = a \inf S$.

    Proof. By hypothesis, $aS \neq \varnothing$ is bounded. By completeness of $\mathbb{R}$, $\inf S$ and $\inf(aS)$ exist. We show $a \inf S = \inf(aS)$ by showing $a \inf S \leq \inf(aS)$ and $a \inf S \geq \inf(aS)$.

    Now, $\inf S \leq s$ for all $s \in S$. Since $a > 0$, $a \inf S \leq as$ for all $s \in S$, so that $a \inf S \leq \inf(aS)$. Similarly, $\inf(aS) \leq as$ for all $s \in S$. Since $a > 0$, $\frac{1}{a} \inf(aS) \leq s$ for all $s \in S$ and so $\frac{1}{a} \inf(aS) \leq \inf S$. As $a > 0$, $\inf(aS) \leq a \inf S$. QED

22. **Short Answer Question:** provide a proof of the following statement.

    Let $S \neq \varnothing$ be a bounded subset of $\mathbb{R}$. Let $b < 0$. Define $bS = \{bs : s \in S\}$. Prove that $\sup(bS) = b \inf S$.

    Proof. By hypothesis, $bS \neq \varnothing$ is bounded. By completeness of $\mathbb{R}$, $\inf S$ and $\sup(bS)$ exist. We show $b \inf S = \sup(bS)$ by showing $b \inf S \geq \sup(bS)$ and $b \inf S \leq \sup(bS)$.

    Now, $\inf S \leq s$ for all $s \in S$. Since $b < 0$, $b \inf S \geq bs$ for all $s \in S$, so that $b \inf S \geq \sup(bS)$. Similarly, $\sup(bS) \geq bs$ for all $s \in S$. Since $b < 0$, $\frac{1}{b} \sup(bS) \leq s$ for all $s \in S$ and so $\frac{1}{b} \sup(bS) \leq \inf S$. As $b > 0$, $\sup(bS) \geq b \inf S$. QED

23. **Short Answer Question:** provide a proof of the following statement.

    Let $A, B \neq \varnothing$ be bounded subsets of $\mathbb{R}$. Define $A + B = \{a + b : a \in A, b \in B\}$. Show that $\sup(A + B) = \sup(A) + \sup(B)$.

    Proof. Let $A, B \neq \varnothing$ be bounded subsets of $\mathbb{R}$. Define $A + B = \{a + b : a \in A, b \in B\}$. Show that $\sup(A + B) = \sup(A) + \sup(B)$.

A and B are bounded and non-empty. By completeness, they have supremums in $\mathbb{R}$, say a_* and b_*, respectively. Then $a_* \geq a$ and $b_* \geq b$ for all $a \in A$, $b \in B$.

The real number $a_* + b_*$ is an upper bound of $A + B$ since $a_* + b_* \geq a + b$ for all $a \in A$, $b \in B$. By completeness of $\mathbb{R}$, $A + B$ has a supremum as it is also not empty. We show that this supremum is indeed $a_* + b_*$.

Let w be an upper bound of $A + B$. Then, $w \geq a + b$ for all $a \in A$ and $b \in B$, or $w - b \geq a$ for all $a \in A$ and $b \in B$. Thus, $w - b$ is an upper bound of A for all $b \in B$, *i.e.* $w - b \geq a_*$ for all $b \in B$. Then $w - a_* \geq b$ for all $b \in B$, so $w - a_*$ is an upper bound of B; hence $w - a_* \geq b_*$. As a result, $w \geq a_* + b_*$, which concludes the proof. QED

24. **Short Answer Question:** provide a proof of the following statement.

If $S \neq \varnothing$ is a bounded subset of $\mathbb{R}$ and $I_S = [\inf S, \sup S]$, show that $S \subseteq I_S$. Moreover, if J is any closed bounded interval containing S, show $I_S \subseteq J$.

Proof. As S is non-empty and bounded, $\sup S$ and $\inf S$ exist by the completeness of $\mathbb{R}$. Since $\inf S \leq s \leq \sup S$ for all $s \in S$, then $\inf S \leq \sup S$ and so the interval $I_s = [\inf S, \sup S]$ is well-defined. Furthermore, the string of inequalities above also shows that $S \subseteq I_S$.

Let $J = [a, b]$ be a closed interval containing S. Then $a \leq s \leq b$ for all $s \in S$. Thus, a is a lower bound and b is an upper bound of S. By definition,

$$a \leq \inf S \leq \sup S \leq b,$$

and so $I_s = [\inf S, \sup S] \subseteq [a, b] = J$. QED

25. **Short Answer Question:** provide a proof of the following statement.

Let $(a_n), (b_n)$ be bounded sequences in $\mathbb{R}$. Show that

$$\limsup_{n\to\infty}(a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n,$$

and show by example that this inequality can be strict.

Proof. Fix $\epsilon > 0$. Then there exists some $N = N(\epsilon)$ so that, for all $m > N$, the following inqualities all hold:

$$\begin{aligned}
\frac{\epsilon}{2} + \limsup_{n\to\infty} a_n \geq a_m \geq -\frac{\epsilon}{2} + \liminf_{n\to\infty} a_n \\
\frac{\epsilon}{2} + \limsup_{n\to\infty} b_n \geq b_m \geq -\frac{\epsilon}{2} + \liminf_{n\to\infty} b_n.
\end{aligned}$$

Adding the left-hand sided inequalities, we get:

$$a_m + b_m \leq \epsilon + \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n.$$

We conclude with our first desired inequality,

$$\limsup_{n\to\infty}(a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n.$$

For the counter-example, consider the sequences $a_n = (-1)^n, b_n = (-1)^{n+1}$. Then $a_n + b_n = 0\ \forall n$, so $\limsup_{n\to\infty}(a_n + b_n) = 0$. However, $\limsup_{n\to\infty} a_n = \limsup_{n\to\infty} b_n = 1$. QED

26. **Short Answer Question:** provide a proof of the following statement.

Let (b_n) be a convergent sequence of real numbers and (a_n) a sequence of real numbers such that $|a_m - a_n| \leq |b_m - b_n|$ for all $n, m \in \mathbb{N}$. Show (a_n) is convergent.

Proof. Fix $\epsilon > 0$. Since (b_n) is convergent, it is also Cauchy. Thus, there exists some $N = N(\epsilon)$ so that for all $m, n > N(\epsilon)$,

$$|b_n - b_m| < \epsilon.$$

But then for all $m, n > N(\epsilon)$, we also have

$$|a_n - a_m| \leq |b_n - b_m| < \epsilon,$$

which means that (a_n) is Cauchy (hence convergent) as well. QED

27. **Short Answer Question:** provide a proof of the following statement.

Fix a sequence (x_n). Assume there exists $\alpha > 0$ so that

$$|x_{n+2} - x_{n+1}| < (1-\alpha)|x_{n+1} - x_n|$$

for all n. Show that (x_n) is Cauchy.

Proof. We begin by showing the following by induction: for all $n \in \mathbb{N}$ with $n \geq 2$,

$$|x_{n+1} - x_n| \leq (1-\alpha)^{n-1}|x_2 - x_1|.$$

The base case, when $n = 2$, is exactly the assumption. To do the induction step, assume this holds for some particular value of n; then

$$\begin{aligned} |x_{n+2} - x_{n+1}| &\leq (1-\alpha)|x_{n+1} - x_n| \\ &\leq (1-\alpha)(1-\alpha)^{n-1}|x_2 - x_1| \\ &= (1-\alpha)^n|x_2 - x_1| \end{aligned}$$

as desired. Note that the first inequality is the assumption of the question, while the second is our induction assumption.

Using this, we calculate for $m, n \in \mathbb{N}$ with $m > n$:

$$
\begin{aligned}
|x_m - x_n| &= |\sum_{j=n}^{m-1}(x_{j+1} - x_j)| \\
&\leq \sum_{j=n}^{m-1} |x_{j+1} - x_j| \\
&\leq \sum_{j=n}^{m-1} (1-\alpha)^{j-1}|x_2 - x_1| \\
&\leq (1-\alpha)^{n-1}|x_2 - x_1| \sum_{j=0}^{\infty}(1-\alpha)^j \\
&= (1-\alpha)^{n-1}\frac{|x_2 - x_1|}{\alpha}.
\end{aligned}
$$

Finally, fix $\epsilon > 0$. There exists N so that for all $n > N$, $(1-\alpha)^{n-1} \leq \frac{\alpha\epsilon}{2|x_2 - x_1|}$. Thus, for $m, n > N$,

$$|x_m - x_n| \leq (1-\alpha)^{n-1}\frac{|x_2 - x_1|}{\alpha} \leq \frac{\epsilon}{2}.$$

This completes the proof. QED

28. **Short Answer Question:** provide a proof of the following statement.

A sequence (a_n) converges to A if and only if every one of its subsequences converges to A.

Proof. We prove the directions separately:

1. **Assume $x_n \to L$.** Consider a strictly increasing function $f : \mathbb{N} \mapsto \mathbb{N}$. Fix $\epsilon > 0$. Since $x_n \to L$, there exists some N so that for all $n > N$, $|x_n - L| < \epsilon$. But for such a value of n, we have $f(n) \geq n > N$, so

$$|x_{f(n)} - L| < \epsilon$$

as well.

2. **Assume $x_{f(n)} \to L$ for any strictly increasing function $f : \mathbb{N} \mapsto \mathbb{N}$.** Then in particular this is true for the function $f(n) = n$, so $x_n \to L$.

This completes the proof. QED

29. **Computations and Applications of Definitions**

Use the definition of the limit of a sequence to show that $\lim_{n\to\infty} \dfrac{3n+1}{2n+5} = \dfrac{3}{2}$.

Proof. Let $\varepsilon > 0$. By the Archimedean property, there is a positive integer $N_\varepsilon > \frac{13}{4\varepsilon}$. Then

$$\left|\frac{3n+1}{2n+5} - \frac{3}{2}\right| = \frac{13}{2(2n+5)} < \frac{13}{4n} < \frac{13}{4N_\varepsilon} < \varepsilon,$$

whenever $n > N_\varepsilon$. QED

30. **Computations and Applications of Definitions**

Use the definition of the limit of a sequence to show that $\lim_{n\to\infty} \dfrac{1}{1+an} = 0$ if $a > 0$.

Proof. Let $\varepsilon > 0$. By the Archimedean property, there is a positive integer $N_\varepsilon > \frac{1}{a\varepsilon}$. Then

$$\left|\frac{1}{1+na} - 0\right| = \frac{1}{1+na} < \frac{1}{na} < \frac{1}{N_\varepsilon a} < \varepsilon,$$

whenever $n > N_\varepsilon$. QED

31. **Computations and Applications of Definitions**

Using the definition, show that $\left(\frac{2n+3}{n}\right)$ is a Cauchy sequence.

Proof. Let $\varepsilon > 0$. According to the Archimedean Property, $\exists N_\varepsilon \in \mathbb{N}$ such that $N_\varepsilon > \frac{6}{\varepsilon}$. Then,

$$\left|\frac{2n+3}{n} - \frac{2m+3}{m}\right| = 3\left|\frac{1}{n} - \frac{1}{m}\right| \leq 3\left(\frac{1}{n} + \frac{1}{m}\right) < 3\cdot\frac{2}{N_\varepsilon} < \varepsilon$$

whenever $m, n > N_\varepsilon$. QED

32. **Computations and Applications of Definitions**

Let (x_n) be the sequence defined by $x_1 = 2$ and $x_n = 2 + \sqrt{x_{n-1}}$. Show (x_n) converges and compute its limit.

Proof. We show (x_n) is increasing and bounded by induction; as a results, (x_n) converges. A quick computation shows $x_2 = 2 + \sqrt{2}$.

Initial case – Clearly, $2 \leq x_1 \leq x_2 \leq 4$.

Induction hypothesis – Suppose $2 \leq x_k \leq x_{k+1} \leq 4$. Then $\sqrt{2} \leq \sqrt{x_k} \leq \sqrt{x_{k+1}} \leq \sqrt{4} = 2$ and so

$$2 \leq 2 + \sqrt{2} \leq 2 + \sqrt{x_k} \leq 2 + \sqrt{x_{k+1}} \leq 2 + 2 = 4,$$

i.e. $2 \leq x_{k+1} \leq x_{k+2} = 4$.

Hence (x_n) is increasing and bounded above by 4: $x = \lim x_n$ exists.

But

$$x = \lim x_n = \lim (2 + \sqrt{x_{n-1}}) = 2 + \lim \sqrt{x_{n-1}} = 2 + \lim \sqrt{x_n} = 2 + \sqrt{x},$$

that is, $x = 2 + \sqrt{x}$. The only solution is $x = 4$. Then, $\lim x_n = 4$. QED

33. **Computations and Applications of Definitions**

Using the $\varepsilon - \delta$ definition or the sequential criterion, determine whether $\lim_{x\to 1} \dfrac{x^2 - x + 1}{x+1}$, for $x \neq -1$ exists (if the limit exists, find it).

Proof. If the limit exists, it must be equal to $\frac{1}{2}$ since the sequence $f(x_n) \to \frac{1}{2}$ if $x_n = 1 + 1/n$.

Let $\varepsilon > 0$ and set $\delta_\varepsilon = \min\{\frac{1}{2}, \frac{3}{2}\varepsilon\}$. Then

$$0 < |x-1| < \delta_\varepsilon \implies |2x-1| < 2 \text{ and } \left|\frac{1}{2(x+1)}\right| < \frac{1}{3}.$$

Thus

$$\left|\frac{x^2-x+1}{x+1} - \frac{1}{2}\right| = \left|\frac{2x-1}{2(x+1)}\right| |x-1| < \frac{2}{3}|x-1| < \frac{2}{3}\delta_\varepsilon < \varepsilon$$

whenever $0 < |x-1| < \delta_\varepsilon$ and $x \in \mathbb{R}$. QED

34. **Computations and Applications of Definitions**

Assume $\lim_{n\to\infty} a_n = 3$. Using the $\varepsilon - \delta$ definition or the sequential criterion, compute $\lim_{n\to\infty} a_n^4$.

Proof. Fix $0 < |\delta| < 1$. We then have the following preliminary calculation:

$$\begin{aligned} |(3+\delta)^4 - 3^4| &= |(4)(3^3)\delta + (6)(3^2)\delta^2 + (4)(3)\delta^3 + \delta^4| \\ &\le (4)(3^3)|\delta| + (6)(3^2)|\delta| + (4)(3)|\delta| + |\delta| \\ &= 175|\delta|. \end{aligned}$$

We now start the body of the proof. Fix $\epsilon > 0$. Since $a_n \to 3$, there exists $N = N(\epsilon)$ so that for all $n > N(\epsilon)$, $|a_n - 3| \le \min(\frac{\epsilon}{200}, \frac{1}{2})$. Using the preliminary calculation, for such a value of n,

$$|a_n^4 - 3^4| \le 175|a_n - 3| \le \frac{\epsilon}{200} < \epsilon.$$

This completes the proof. QED

35. **Computations and Applications of Definitions**

Using the $\varepsilon-\delta$ definition or the sequential criterion, determine whether $\lim_{x\to 0}\operatorname{sgn}(\sin(1/x))$ exists, where

$$\operatorname{sgn}(y)=\begin{cases}1 & \text{if } y>0\\ 0 & \text{if } y=0\\ -1 & \text{if } y<0\end{cases}.$$

If the limit exists, find it. For this question, you may assume that the regular properties of the sin function are known.

Proof. Let $(x_n)=\left(\frac{2}{(2n+1)\pi}\right)$. Then $x_n\to 0$, $x_n\neq 0$ for all $n\in\mathbb{N}$ and

$$\operatorname{sgn}\left(\sin\left(\frac{1}{x_n}\right)\right)=\operatorname{sgn}\left((-1)^n\right)=(-1)^n,$$

which does not converge. Hence $\lim_{x\to 0}\operatorname{sgn}\left(\sin\left(\frac{1}{x}\right)\right)$ does not exist.

36. **Long Answer Question:**

Show that there exists a positive real number y such that $y^4=2$.

Proof. Consider the set $S=\{s\mathbb{R}^+ : s^4<2\}$, where $\mathbb{R}^+$ denotes the set of positive real numbers. This set in not empty as $1\in S$. Furthermore, S is bounded above by 2. Indeed, if $t\geq 2$, then $t^4\geq 16>2$, whence $t\notin S$. By completeness of $\mathbb{R}$, $x=\sup S\geq 1$ exists. It will be enough to show that neither $x^4<2$ and $x^4>2$ can hold. The only remaining possibility will be that $x=\sqrt[4]{2}$.

1. If $x^4<2$, then $\frac{4x^3+6x^2+4x+1}{2-x^4}>0$. By the Archimedean property, there is an integer $n>0$ such that $\frac{4x^3+6x^2+4x+1}{2-x^4}<n$. By re-arranging the terms, we get

$$0<\frac{1}{n}(4x^3+6x^2+4x+1)<2-x^4.$$

Then

$$\begin{aligned}\left(x+\frac{1}{n}\right)^4 &= x^4+\frac{4x^3}{n}+\frac{6x^2}{n^2}+\frac{4x}{n^3}+\frac{1}{n^4}\leq x^4+\frac{4x^3}{n}+\frac{6x^2}{n}+\frac{4x}{n}+\frac{1}{n}\\ &= x^4+\frac{1}{n}(4x^3+6x^2+4x+1)<x^4+2-x^4=2.\end{aligned}$$

Since $(x+\frac{1}{n})^4<2$, $x+\frac{1}{n}\in S$. But $x<x+\frac{1}{n}$; x is then not an upper bound which contradicts the fact that $x=\sup S$. Then $x^4\not< 2$.

2. If $x^4>2$, then $\frac{4x^3+4x}{x^4-2}>0$. By the Archimedean property, there is an integer $n>0$ such that $\frac{4x^3+4x}{x^4-2}<n$. By re-arranging the terms, we get

$$0>-\frac{(4x^3+4x)}{n}>2-x^4.$$

Then

$$\left(x-\frac{1}{n}\right)^4 = x^4 - \frac{4x^3}{n} + \frac{6x^2}{n^2} - \frac{4x}{n^3} + \frac{1}{n^4} \geq x^4 - \frac{4x^3}{n} - \frac{4x}{n^3} \geq x^3 - \frac{4x^3}{n} - \frac{4x}{n}$$
$$= x^3 - \frac{1}{n}(4x^3 + 4x) > x^4 + 2 - x^4 = 2.$$

Since $(x - \frac{1}{n})^4 > 2$, $x - \frac{1}{n}$ is an upper bound of S. But $x > x - \frac{1}{n}$; x can not be the supremum, which is a contradiction. Thus $x^4 \not> 2$ or, equivalently, $x = \sqrt[4]{2}$ is a positive real number. QED

37. **Long Answer Question:**

Fix a sequence (x_n) that converges to $L \in \mathbb{R}$. Define $y_n = n^{-1}\sum_{k=1}^{n} x_n$. Show directly that (y_n) converges to L as well.

Proof. Consider a sequence (x_n) with $x_n \to L$. Let $y_n = n^{-1}\sum_{k=1}^{n} x_n$. We will show that $y_n \to L$ as well.

Fix $\delta > 0$. Since $x_n \to L$, there exists $N = N(\delta)$ so that for all $n > N(\delta)$, $|x_n - L| < \frac{\delta}{4}$. Using such a value of n, we have:

$$\begin{aligned}
|y_n - L| &= |n^{-1}\sum_{k=1}^{n} x_n - L| \\
&\leq n^{-1}\sum_{k=1}^{N} |x_n - L| + n^{-1}\sum_{k=N+1}^{n} |x_n - L| \\
&\leq n^{-1}\sum_{k=1}^{N} |x_n - L| + n^{-1}\sum_{k=N+1}^{n} \frac{\delta}{4} \\
&\leq n^{-1}\sum_{k=1}^{N} |x_n - L| + \frac{\delta}{4}.
\end{aligned}$$

Define $A(\delta) = \sum_{k=1}^{N(\delta)} |x_n - L|$. Summarizing the above, we have shown:

$$|y_n - L| \leq \frac{A(\delta)}{n} + \frac{\delta}{4}.$$

$$|y_n - L| \leq \frac{A(\epsilon)}{n} + \frac{\epsilon}{4} \leq \frac{\epsilon}{4} + \frac{\epsilon}{4} < \epsilon.$$

This completes the proof. QED

38. **Long Answer Question:**

Consider the Cauchy condition, with $|x_m - x_n|$ replaced by the adjacent-pair distance $|x_{n+1} - x_n|$. Prove or disprove: this condition also implies convergence.

Proof. This condition does *not* imply convergence. To see this, we will construct a counterexample. Define $x_n = \sum_{k=1}^{n} \frac{1}{\sqrt{k}}$. We now check that (i) this sequence satisfies the modified condition but (ii) this sequence does not converge. Since all Cauchy sequences converge, this will imply that (x_n) is not Cauchy.

1. We can immediately see that $|x_{n+1} - x_n| = \frac{1}{\sqrt{n+1}}$. Fix $\epsilon > 0$. For all $n > 100\lceil \epsilon^{-2} \rceil$,

$$|x_{n+1} - x_n| \leq \frac{1}{\sqrt{n}} \leq \frac{\epsilon}{10} < \epsilon.$$

   Thus, (x_n) satisfies the modified Cauchy condition.

2. Next, fix $m \in \mathbb{N}$. We (sloppily) calculate:

$$\begin{aligned}
|x_{4^m} - x_{4^{m-1}}| &= \sum_{k=4^{m-1}+1}^{4^m} \frac{1}{\sqrt{k}} \\
&\geq \sum_{k=4^{m-1}+1}^{4^m} \frac{1}{\sqrt{4^{m-1}+1}} \\
&\geq (4^m - 4^{m-1} - 1)2^{-m} \\
&\geq 4^{m-1}2^{-m} = 2^{m-1}.
\end{aligned}$$

   Since (x_n) is clearly monotone increasing, we have the telescoping sum inequality:

$$\begin{aligned}
x_{4^m} &= \sum_{j=0}^{m-1} (x_{4^{m-j}} - x_{4^{m-j-1}}) \\
&\geq \sum_{j=0}^{m-1} 2^{m-j-1} \geq m.
\end{aligned}$$

   Since the subsequence (x_{4^m}) is bounded from below by a sequence that diverges to infinity, the subsequence (x_{4^m}) also diverges to infinity.

This completes the proof. QED