MAT 2125 – Homework 3 – Solutions

(due at midnight on March 01, in Brightspace)

1 Norms

1. Suppose $\|\cdot\|$ and $\|\cdot\|'$ are two arbitrary norms on $\mathbf{R}^d$. Prove that $\|\cdot\|''$, defined by

$$\|\mathbf{x}\|'' = \|\mathbf{x}\| + \|\mathbf{x}\|', \quad \mathbf{x} \in \mathbb{R}^d,$$

is also a norm on $\mathbb{R}^d$.

Proof: For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$, $\lambda \in \mathbb{R}$, we have

- $\mathbf{x} = 0 \iff \|\mathbf{x}\| = 0$ and $\|\mathbf{x}\|' = 0 \iff \|\mathbf{x}\| + \|\mathbf{x}\|' = 0 \iff \|\mathbf{x}\|'' = 0$;
- $\|\lambda\mathbf{x}\|'' = \|\lambda\mathbf{x}\| + \|\lambda\mathbf{x}\|' = |\lambda| \cdot \|\mathbf{x}\| + |\lambda| \cdot \|\mathbf{x}\|' = |\lambda| \cdot (\|\mathbf{x}\| + \|\mathbf{x}\|') = |\lambda| \cdot \|\mathbf{x}\|''$, and
- $\|\mathbf{x}+\mathbf{y}\|'' = \|\mathbf{x}+\mathbf{y}\| + \|\mathbf{x}+\mathbf{y}\|' \leq \|\mathbf{x}\| + \|\mathbf{y}\| + \|\mathbf{x}\|' + \|\mathbf{y}\|' = \|\mathbf{x}\|'' + \|\mathbf{y}\|''$.

Since $\|\cdot\|''$ satisfies N1, N2, and N3, it is a norm. ■

2. A function $\|\cdot\|$ is called a *seminorm* if it satisfies all of the norm properties, *except possibly* N1 (norms are also seminorms). Let $\|\cdot\|_1, \ldots, \|\cdot\|_k$ be a collection of seminorms on $\mathbb{R}^d$. Assume that, for all $\mathbf{x} \in \mathbb{R}^d \setminus \{0\}$, there exists $i \in \{1, \ldots, k\}$ so that $\|\mathbf{x}\|_i > 0$. Show that $\|\cdot\| = \sum_{i=1}^{k} \|\cdot\|_i$ is a norm.

Proof: N2, N3 are exactly as in the previous question (note that the proof of the N2 property only uses the N2 property of the constituent norms, and similarly for the N3 property). We now prove N1.

If $\mathbf{x} = \mathbf{0}$, then $\|\mathbf{x}\| = \|0 \cdot \mathbf{x}\| = |0|\|\mathbf{x}\| = 0$, applying N2 to the second equality. Conversely, if $\mathbf{x} \neq 0$, then by assumption there exists $i \in \{1, \ldots, k\}$ such that $\|\mathbf{x}\|_i > 0$; consequently

$$\|\mathbf{x}\| = \|\mathbf{x}\|_i + \sum_{j \neq i=1}^{k} \|\mathbf{x}\|_j \neq 0.$$

Combined, these two results show that $\|\mathbf{x}\| = 0 \iff \mathbf{x} = \mathbf{0}$.

3. Suppose $\{\mathbf{x}_n\}_{n=1}^{\infty}$ is a sequence in $\mathbb{R}^d$ converging to $\mathbf{y}$. Prove that $\lim_{n\to\infty} \|\mathbf{x}_n\| = \|\mathbf{y}\|$.

Proof: By the triangle inequality, we have

$$\|\mathbf{x}_n\| \leq \|\mathbf{x}_n - \mathbf{y}\| + \|\mathbf{y}\| \implies \|\mathbf{x}_n\| - \|\mathbf{y}\| \leq \|\mathbf{x}_n - \mathbf{y}\| \quad \text{and}$$
$$\|\mathbf{y}\| \leq \|y - \mathbf{x}_n\| + \|\mathbf{x}_n\| \implies \|\mathbf{y}\| - \|\mathbf{x}_n\| \leq \|y - \mathbf{x}_n\| = \|\mathbf{x}_n - \mathbf{y}\| \implies -\|\mathbf{x}_n - \mathbf{y}\| \leq \|\mathbf{x}_n\| - \|\mathbf{y}\|.$$

Thus

$$-\|\mathbf{x}_n - \mathbf{y}\| \leq \|\mathbf{x}_n\| - \|\mathbf{y}\| \leq \|\mathbf{x}_n - \mathbf{y}\|.$$

Since $\|\mathbf{x}_n - \mathbf{y}\| \to 0$, it follows from the Squeeze Theorem that $\|\mathbf{x}_n\| - \|\mathbf{y}\| \to 0$, and so $\|\mathbf{x}_n\| \to \|\mathbf{y}\|$. ■

2 Closed and Open Sets

1. Fix an n by n matrix M. Show that, for any $C \geq 0$, $S = \{x \in \mathbb{R}^n : x^t M x \leq C\}$ is a closed set.

Proof: Consider $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. By the triangle inequality, we have

$$\|\mathbf{x}^\top M\mathbf{x} - \mathbf{y}^\top M\mathbf{y}\| \leq \|\mathbf{x}^\top M\mathbf{x} - \mathbf{x}^\top M\mathbf{y}\| + \|\mathbf{x}^\top M\mathbf{y} - \mathbf{y}^\top M\mathbf{y}\|.$$

But

$$\begin{aligned}\|\mathbf{x}^\top M\mathbf{x} - \mathbf{x}^\top M\mathbf{y}\| &= \|\mathbf{x}^\top M(\mathbf{x}-\mathbf{y})\| \le \|\mathbf{x}^\top M\|_\infty \|\mathbf{x}-\mathbf{y}\| \\ &\le n^2 \cdot \max\{|\mathbf{x}[i]| \mid i = 1,\dots,n\} \cdot \max\{|M[i,j]| \mid i,j = 1,\dots,n\} \|\mathbf{x}-\mathbf{y}\|.\end{aligned}$$

Similarly,

$$\|\mathbf{x}^\top M\mathbf{y} - \mathbf{y}^\top M\mathbf{y}\| \le n^2 \cdot \max\{|\mathbf{y}[i]| \mid i = 1,\dots,n\} \cdot \max\{|M[i,j]| \mid i,j = 1,\dots,n\} \|\mathbf{x}-\mathbf{y}\|.$$

Let $A = \max\{1, 2n^2, |\mathbf{x}[1]|, \dots, |\mathbf{x}[n]|, |\mathbf{y}[1]|, \dots, |\mathbf{y}[n]|, |M[1,1]|, \dots, |M[n,n]\} < \infty$. Then

$$\|\mathbf{x}^\top M\mathbf{x} - \mathbf{y}^\top M\mathbf{y}\| \le A\|\mathbf{x}-\mathbf{y}\|.$$

It remains to prove that S is closed. We will do so by proving that $S^{\complement}$ is open. Let $\mathbf{x} \in S^{\complement}$. By definition, $\delta = \mathbf{x}^\top M\mathbf{x} - C > 0$. Set $\varepsilon = \frac{\delta}{3A}$ and pick $\mathbf{y}$ such that $\|\mathbf{x}-\mathbf{y}\| < \varepsilon$.

From the reverse triangle inequality and the previous remarks, we have

$$\begin{aligned}\|\mathbf{y}^\top M\mathbf{y}\| &\ge \|\mathbf{x}^\top M\mathbf{x}\| - \|\mathbf{x}^\top M\mathbf{x} - \mathbf{y}^\top M\mathbf{y}\| > C + \delta - \|\mathbf{x}^\top M\mathbf{x}\| - \|\mathbf{x}^\top M\mathbf{x} - \mathbf{y}^\top M\mathbf{y}\| \\ &\ge C + \delta - A\|\mathbf{x}-\mathbf{y}\| \ge C + \delta - A\frac{\delta}{3A} = C + \frac{2}{3}\delta > C.\end{aligned}$$

Thus $\mathbf{y} \in S^{\complement}$, which shows that there is an open ball around $\mathbf{x}$ that stays completely within $S^{\complement}$, i.e. S is closed. ■

2. Consider a sequence (a_n) with two distinct accumulation points. Show that the sequence does not have a limit.

   Proof: We prove this by contradiction. First, assume that the sequence has a limit L. Denote by A, B two distinct accumulation points of the sequence, and assume (without loss of generality) that $|L - A| \le |L - B|$, so that
   $$|A - B| \le |A - L| + |L - B| \le 2|L - B|.$$

   Set $\varepsilon = \frac{|B-a|}{10}$. Since the limit of the sequence (a_n) is L, there exists some $N \in \mathbb{N}$ such that for all $n > N$, $|a_n - L| < \frac{|B-A|}{10}$. Then

   $$|B - a_n| \ge |B - L| - |L - a_n| \ge \frac{1}{2}|A - B| - \frac{1}{10}|A - B| = \frac{2}{5}|A - B|.$$

   In particular, this means that there are only finitely many points of (a_n) within distance $\frac{|B-A|}{10}$ of B, so B cannot be an accumulation point of (a_n). This is a contradiction, and so the assumption that (a_n) converges is false. ■

3. Show that there exists a sequence (a_n) whose set of accumulation points is exactly the interval $[0, 1]$. You don't need to write down an explicit formula for the sequence - you just need to show that such a sequence exists.

   Proof: Let (a_n) be a list of the countable set $\mathbb{Q} \cap [0, 1]$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, it follows immediately that, as a set, $\{a_n\}$ is dense in $\mathbb{R} \cap [0, 1]$, that is, in the neighbourhood of any point in $[0, 1]$, there are infinitely many rational numbers. In particular, each point of $[0, 1]$ is an accumulation point of (a_n). ■

4. Does there exists a sequence (a_n) whose set of accumulation points is exactly the interval $(0, 1)$?

   Proof: As we will see below, the set of accumulation points of a sequence is necessarily closed; since $(0, 1)$ is open, no sequence's accumulation points can be exactly $(0, 1)$.

Let $(\mathbf{a}_n)$ be a sequence with a set of accumulation points S. Consider $\mathbf{x} \in S^{\complement}$. Then $\exists r_x\ 0$ such that $|B(\mathbf{x}, r_x) \cap \{\mathbf{a}_n\}| < \infty$. Setting $\varepsilon = \frac{r}{3}$, we note that for all $\mathbf{y}$ with $\|\mathbf{x} - \mathbf{y}\| \leq \varepsilon$, we have $B(\mathbf{y}, \varepsilon) \subseteq B(\mathbf{x}, r_x)$. Thus

$$|B(\mathbf{y}, \varepsilon) \cap \{\mathbf{a}_n\}| \leq |B(\mathbf{x}, r_x) \cap \{\mathbf{a}_n\}| < \infty;$$

thus none of the $\mathbf{y}$ in $\|\mathbf{x} - \mathbf{y}\| \leq \varepsilon$ can be an accumulation point of $(\mathbf{a}_n)$, so $\mathbf{y} \in S^{\complement}$. As a result, $S^{\complement}$ is open, so S is closed. ∎

5. Prove that for all $\varepsilon > 0$ there exists a collection of open sets $(a_1, b_1), (a_2, b_2), \ldots$ satisfying the following two properties:

$$\bigcup_{n \in \mathbb{N}} (a_n, b_n) \supseteq \mathbb{Q} \quad \text{and} \quad \sum_{n \in \mathbb{N}} (b_n - a_n) < \varepsilon.$$

Hint: Recall that $\sum_{n \in \mathbb{N}} 2^{-n} = 1$.

Proof: Let (q_n) be an enumeration of $\mathbf{Q}$ and fix $\varepsilon > 0$. Define

$$(a_n, b_n) = (q_n - \varepsilon 2^{-n-4}, q_n + \varepsilon 2^{-n-4}).$$

It is clear that $q_n \in (a_n, b_n)$ for all n, so

$$\mathbb{Q} = \bigcup_{n \in \mathbb{N}} \{q_n\} \subseteq \bigcup_{n \in \mathbb{N}} (a_n, b_n).$$

Additionally,

$$\sum_{n \in \mathbb{N}} (b_n - a_n) = \sum_{n \in \mathbb{N}} 2\varepsilon \cdot 2^{-n-4} = \frac{\varepsilon}{8} < \varepsilon,$$

which completes the proof.[1] ∎

3 Compact Sets

1. Show that a finite union of compact sets is compact.

Proof: Suppose $K_1, \ldots, K_n$ are compact subsets of $\mathbb{R}^d$. Then, by the Heine–Borel Theorem, they are closed and bounded. Since they are bounded, for each $i \in \{1, \ldots, d\}$, there exists $M_i > 0$ such that

$$\|x\| \leq M_i \quad \forall\, x \in K_i.$$

Setting $M = \max\{M_1, \ldots, M_d\}$, we then see that

$$\|x\| \leq M \quad \forall\, x \in K_1 \cup \cdots \cup K_d.$$

So the union is bounded. It is also closed as the finite union of closed sets is closed, by a propositions seen in class. Thus it is compact by the Heine-Borel Theorem. [2] ∎

2. Show that an arbitrary intersection of compact sets is compact.

Proof: Suppose K_α, $\alpha \in I$, are compact sets. Then they are closed and bounded by the Heine-Borel Theorem. As the arbitrary intersection of closed sets is closed, $\bigcap_{\alpha \in I} K_\alpha$ is also closed. Now fix some $\beta \in I$. Since

$$\bigcap_{\alpha \in I} K_\alpha \subseteq K_\beta,$$

the intersection $\bigcap_{\alpha \in I} K_\alpha$ is bounded, since K_β is. So the intersection is closed and bounded, hence compact. ∎

[1] We will discuss the convergence of series later in the course; for now, we assume that infinite geometric series can be evaluated as in calculus.

[2] *Alternate solution*: Suppose $K_1, \ldots, K_n$ are compact subsets of $\mathbb{R}^d$. Let $\mathcal{U}$ be an open cover of the union. Then, for each $i \in \{1, \ldots, n\}$, there is a finite subcover $\mathcal{U}_i$ of K_i. Then $\mathcal{U}_1 \cup \cdots \cup \mathcal{U}_d$ is a finite subcover of $K_1 \cup \cdots \cup K_d$.