MAT 2125 – Homework 4 – Solutions

(due at midnight on March 26, in Brightspace)

1 Continuous Functions

1. Define $g : \mathbb{R} \to \mathbb{R}$ by

$$g(x) = \begin{cases} \frac{(-1)^n}{n} & \text{if } x = 1/n \text{ for some } n \in \mathbb{N}, \\ 0 & \text{otherwise} \end{cases}$$

Prove that g is continuous at 0.

Proof: Let $\varepsilon > 0$. Set $\delta_\varepsilon = \varepsilon$. Then,

$$\left| \frac{1}{n} - 0 \right| < \delta \implies |g(1/n) - g(0)| = 1/n = |1/n| < \delta = \varepsilon.$$

So g is continuous at 0. ■

2. Assume that the temperature distribution on the Earth's equator is continuous. Show that there are, at any time, antipodal points on the Earth's equator with the same temperature.

Proof: At a given moment, let the temperature on the Earth's equator be given by a continuous function $f : [0, 1] \to \mathbb{R}$ such that $f(0) = f(1)$.

The coordinate x could represent the Eastward distance from Quito, Ecuador, say, as a fraction of the Earth's circumference at the equator. The antipode of a point x in $[0, 1]$ is

$$a(x) = \begin{cases} x + \frac{1}{2} & \text{if } x \in [0, \frac{1}{2}] \\ x - \frac{1}{2} & \text{if } x \in [\frac{1}{2}, 1] \end{cases}$$

Consider the function $g : [0, \frac{1}{2}] \to \mathbb{R}$ defined by

$$g(x) = f(x) - f(a(x)).$$

As f and a are continuous, g is also continuous, being the difference of the continuous function f and the composition of the continuous functions f and a.

Now, note that $g(0) = f(0) - f(a(0)) = f(0) - f\left(\frac{1}{2}\right)$ and

$$g\left(\tfrac{1}{2}\right) = f\left(\tfrac{1}{2}\right) - f\left(a\left(\tfrac{1}{2}\right)\right) = f\left(\tfrac{1}{2}\right) - f(1) = f\left(\tfrac{1}{2}\right) - f(0).$$

Thus $g(0)g(\frac{1}{2}) = -(g(0))^2 < 0$; by the Intermediate Value Theorem,

$$\exists c \in (0, \tfrac{1}{2}) \text{ s.t. } g(c) = 0 \implies f(c) = f(a(c)),$$

which completes the proof. ■

3. Suppose $f : \mathbb{R}^d \to \mathbb{R}^m$. The *pre-image* of a subset $B \subseteq \mathbb{R}^m$ under f is

$$f^{-1}(B) = \{\mathbf{a} \in A : f(\mathbf{a}) \in B\}.$$

Prove that f is continuous if and only if the pre-image of every open subset of $\mathbb{R}^m$ is an open subset of $\mathbb{R}^d$. (It is also true if "open" is replaced by "closed", but we will not ask you to prove this.)

Hint: what is the definition of continuity for functions $f : \mathbb{R}^d \to \mathbb{R}^m$?

Proof: Suppose f is continuous and let $B \subseteq \mathbb{R}^m$ be open. Choose $\mathbf{a} \in f^{-1}(B)$. Thus $f(\mathbf{a}) \in B$. Since B is open, there exists $\varepsilon > 0$ such that

$$B(f(\mathbf{a}), \varepsilon) \subseteq B.$$

Since f is continuous, there exists $\delta > 0$ such that

$$f(B(\mathbf{x}, \delta)) \subseteq B(f(\mathbf{x}), \varepsilon) \subseteq B.$$

Thus $B(\mathbf{a}, \delta) \subseteq f^{-1}(B)$. So $f^{-1}(B)$ is open.

Now suppose that the pre-image of every open subset of $\mathbb{R}^m$ is open. Let $\mathbf{a} \in \mathbb{R}^d$ and $\varepsilon > 0$. Then $B(f(\mathbf{a}), \varepsilon)$ is an open subset of $\mathbb{R}^m$. Therefore, by assumption, $f^{-1}(B(f(\mathbf{a}), \varepsilon))$ is open. Since $a \in f^{-1}(B(f(\mathbf{a}), \varepsilon))$, this means that there exists $\delta > 0$ such that

$$B(\mathbf{a}, \delta) \subseteq f^{-1}(B(f(\mathbf{a}), \varepsilon)) \implies f(B(\mathbf{a}, \delta)) \subseteq B(f(\mathbf{a}), \varepsilon).$$

Therefore f is continuous at $\mathbf{a}$. Since $\mathbf{a}$ was arbitrary, f is continuous.[1] ■

4. A function $f : A \to \mathbb{R}$ is said to be *Lipschitz* if there is a positive number M such that

$$|f(x) - f(y)| \leq M|x - y| \quad \forall\, x, y \in A.$$

Show that a Lipschitz function must be uniformly continuous, but that uniformly continuous functions do not have to be Lipschitz.

Hint: for the second statement, consider the function $g : [0, 1] \to \mathbb{R}, \quad g(x) = \sqrt{x}$.

Proof: We will prove the statement in the general multi-dimensional case. The one-dimensional case will then simply be a special case of the more general result.

Suppose f is Lipschitz and $\mathbf{a} \in A$. Let $\varepsilon > 0$. Set $\delta = \varepsilon/M$. Then

$$\|\mathbf{x} - \mathbf{y}\| < \delta \implies \|f(\mathbf{x}) - f(\mathbf{y})\| \leq M\|\mathbf{x} - \mathbf{y}\| < M\varepsilon/M = \varepsilon.$$

Thus f is uniformly continuous.

The function g is continuous on the compact interval $[0, 1]$, hence it is uniformly continuous by a Theorem seen in class (continuous functions on compact sets are uniformly continuous). Assume that g is Lipschitz. Then $\exists M > 0$ such that

$$|h(x) - h(0)| \leq M|x - 0| \quad \forall\, x \in [0, 1] \implies \sqrt{x} \leq Mx \quad \forall\, x \in [0, 1]. \implies M \geq \frac{1}{\sqrt{x}} \quad \forall\, x \in [0, 1].$$

This contradicts the fact that $1/\sqrt{x} \to \infty$ as $x \to 0^+$. Hence, g cannot be Lipschitz. ■

[1] The pre-image of closed sets by a continuous function is also closed. Note that

$$f^{-1}(B)^{\complement} = \{\mathbf{a} \in A : f(\mathbf{a}) \notin B\} = \{\mathbf{a} \in A : f(\mathbf{a}) \in B^{\complement}\} = f^{-1}\left(B^{\complement}\right).$$

Hence

$$\begin{aligned} f \text{ is continuous} &\iff f^{-1}(B) \text{ is open for all open } B \iff f^{-1}(B)^{\complement} \text{ is closed for all open } B \\ &\iff f^{-1}\left(B^{\complement}\right) \text{ is closed for all open } B \iff f^{-1}(C) \text{ is open for all closed } C, \end{aligned}$$

where in the last if and only if statement we let $C = B^{\complement}$ (so C is closed if and only if B is open).

2 Differentiation

1. Let $a \in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x^2 & \text{if } x \geq 0, \\ ax & \text{if } x < 0. \end{cases}$$

For which values of a is f differentiable at $x = 0$? For which values of a is f continuous at $x = 0$?

Proof: We have

$$f'_+(0) = \lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^+} \frac{x^2}{x} = \lim_{x \to 0^+} x = 0$$

and

$$f'_-(0) = \lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^-} \frac{ax}{x} = \lim_{x \to 0^+} a = a.$$

Thus, f is differentiable at $x = 0$ if and only if $a = 0$.

Since both x^2 and ax are continuous functions, we have

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x^2 = 0 = f(0) = 0 = \lim_{x \to 0^-} ax = \lim_{x \to 0^-} f(x)$$

and the the function f is continuous at $x = 0$ for all values of a. ■

2. Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Show that f is Lipschitz if and only if f' is bounded on (a, b).

Hint: Apply the Mean Value Theorem to f on $[x, y] \subseteq [a, b]$ to show one of the directions.

Proof: Suppose that f satisfies the Lipschitz condition on $[a, b]$ with constant M. Then, for all $x_0 \in (a, b)$, we have

$$\left| \frac{f(x) - f(x_0)}{x - x_0} \right| \leq M \qquad \forall\, x \in (a, b) \setminus \{x_0\}.$$

Thus

$$|f'(x_0)| = \left| \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \right| = \lim_{x \to x_0} \left| \frac{f(x) - f(x_0)}{x - x_0} \right| \leq M,$$

where we used the fact that the absolute value function is continuous to pull the limit out of the absolute value. So the derivative of f is bounded on (a, b).

Now assume that $|f'(x)| \leq M$ for all $x \in (a, b)$. Let $x, y \in [a, b]$, $x < y$. Applying the Mean Value Theorem to f on the interval $[x, y]$ gives the existence of $c \in (x, y)$ such that

$$\frac{f(y) - f(x)}{y - x} = f'(c).$$

Thus

$$\left| \frac{f(x) - f(y)}{x - y} \right| \leq M \implies |f(x) - f(y)| \leq M|x - y|.$$

This completes the proof. ■

3. If $x > 0$, show $1 + \frac{1}{2}x - \frac{1}{8}x^2 \leq \sqrt{1 + x} \leq 1 + \frac{1}{2}x$.

Proof: Let $x_0 = 0$ and $f(x) = \sqrt{1 + x}$. According to Taylor's theorem, $f(x) = P_1(x) + R_1(x)$ and

$f(x) = P_2(x) + R_2(x)$, where

$$P_1(x) = f(x_0) + f'(x_0)(x - x_0) = \sqrt{1+0} + \frac{1}{2\sqrt{1+0}}x = 1 + \frac{1}{2}x$$

$$P_2(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2 = \sqrt{1+0} + \frac{1}{2\sqrt{1+0}}x - \frac{1}{8\sqrt[3]{1+0}}x^2 = 1 + \frac{1}{2}x - \frac{1}{8}x^2$$

$$R_1(x) = \frac{f''(c_1)}{2}(x - x_0)^2 = -\frac{1}{8\sqrt[3]{1+c_1}}x^2, \quad \text{for some } c_1 \in [0, x]$$

$$R_2(x) = \frac{f'''(c_2)}{6}(x - x_0)^3 = \frac{3}{48\sqrt[5]{1+c_2}}x^3, \quad \text{for some } c_2 \in [0, x].$$

But $R_1(x) \leq 0$ and $R_2(x) \geq 0$ for all $x > 0$. Hence, $P_2(x) \leq f(x) \leq P_1(x)$, which completes the proof. ■

3 Riemann Integral

1. Using the definition of Riemann-integrability, show that $h : [a, b] \to \mathbb{R}$ defined by $h(x) = 2x + 1$ is Riemann-integrable on $[a, b]$, $b > a \geq 0$, and that that the Riemann integral of h on $[a, b]$ is $b^2 - a^2 + b - a$.

   Warning: you cannot use the rules of integration from calculus.

   Proof: Let $n \in \mathbb{N}$ and $P_n = \left\{x_i = a + \frac{b-a}{n} \cdot i \mid i = 0, \ldots, n\right\}$ be the partition of $[a, b]$ into n equal segments. Set $m_i = \inf\{f(x) \mid x \in [x_{i-1}, x_i]\}$, for $i = 1, \ldots, n$. With this notation, we have

$$L(P_n; h) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}) = \frac{b-a}{n}\sum_{i=1}^{n} m_i.$$

   But $h'(x) = 2 \geq 0$ when $x \geq 0$, and so h is increasing on $[a, b]$.

   Consequently, for $i = 1, \ldots, n$, we have

$$m_i = 2x_{i-1} + 1 = 2\left(a + \tfrac{b-a}{n}(i-1)\right) + 1 = (2a+1) + \tfrac{2(b-a)}{n}(i-1).$$

   The lower sum of h associated to P_n is thus

$$L(P_n; h) = \frac{b-a}{n}\sum_{i=1}^{n}\left((2a+1) + \frac{2(b-a)}{n}(i-1)\right) = \frac{n(b-a)(2a+1)}{n} + \frac{2(b-a)^2}{n^2}\sum_{i=1}^{n}(i-1)$$
$$= (b-a)(2a+1) + \frac{2(b-a)^2}{n^2}\cdot\frac{n(n-1)}{2} = (b-a)(2a+1) + (b-a)^2\left(1 - \frac{1}{n}\right).$$

   But for the lower sum of h on $[a, b]$, we have

$$L(h) = \sup\{L(P; h) \mid P \in \mathcal{P}([a, b])\} \geq \sup_{n\in\mathbb{N}}\{L(P_n; h)\} = \sup_{n\in\mathbb{N}}\left\{(b-a)(2a+1) + (b-a)^2\left(1 - \frac{1}{n}\right)\right\}$$
$$= \lim_{n\to\infty}\left[(b-a)(2a+1) + (b-a)^2\left(1 - \frac{1}{n}\right)\right] = (b-a)(2a+1) + (b-a)^2 = b^2 - a^2 + b - a.$$

   Similarly, since h is increasing we have, for $i = 1, \ldots, n$,

$$M_i = \sup\{f(x) \mid x \in [x_{i-1}, x_i]\} = 2x_i + 1 = 2\left(a + \tfrac{b-a}{n}\cdot i\right) + 1 = (2a+1) + \tfrac{2(b-a)}{n}\cdot i,$$

   from which we derive $U(P_n; h) = (b-a)(2a+1) + (b-a)^2\left(1 + \frac{1}{n}\right)$, so that

$$U(h) \leq \inf_{n\in\mathbb{N}}\{U(P_n; h)\} = \lim_{n\to\infty} U(P_n; h) = b^2 - a^2 + b - a.$$

   Thus $b^2 - a^2 + b - a \leq L(h) \leq U(h) \leq b^2 - a^2 + b - a$ and so $L(h) = U(h) = \int_a^b h = b^2 - a^2 + b - a$. ■

2. Prove *Riemann's Criterion* for a bounded function $f : [a,b] \to \mathbb{R}$, namely: f is Riemann-integrable over $[a,b]$ if and only if $\forall \varepsilon > 0$, $\exists P_\varepsilon$ a partition of $[a,b]$ such that the lower sum $L(P_\varepsilon; f)$ and the upper sum $U(P_\varepsilon; f)$ of f corresponding to P_ε satisfy $U(P_\varepsilon; f) - L(P_\varepsilon; f) < \varepsilon$.

   Proof: If f is Riemann-integrable, then $L(f) = U(f) = \int_a^b f$. Let $\varepsilon > 0$. Since $\int_a^b f - \frac{\varepsilon}{2}$ is not an upper bound of $\{L(P; f) \mid P \text{ a partition of } [a,b]\}$, there exists a partition P_1 such that

   $$\int_a^b f - \frac{\varepsilon}{2} < L(P_1; f) \leq \int_a^b f.$$

   Using a similar argument, there exists a partition P_2 such that

   $$\int_a^b f + \frac{\varepsilon}{2} \geq U(P_2; f) > \int_a^b f.$$

   Set $P_\varepsilon = P_1 \cup P_2$. Then P_ε is a refinement of P_1 and P_2, so

   $$\int_a^b f - \frac{\varepsilon}{2} < L(P_1; f) \leq L(P_\varepsilon; f) \leq U(P_\varepsilon; f) \leq U(P_2; f) < \int_a^b f + \frac{\varepsilon}{2} \implies U(P_\varepsilon; f) - L(P_\varepsilon; f) < \varepsilon.$$

   Conversely, let $\varepsilon > 0$ and let P_ε be a partition of $[a,b]$ such that $U(P_\varepsilon; f) - L(P_\varepsilon; f) < \varepsilon$. Since $U(f) \leq U(P_\varepsilon; f)$ and $L(f) \geq L(P_\varepsilon; f)$, then

   $$0 \leq U(f) - L(f) \leq U(P_\varepsilon; f) - L(P_\varepsilon; f) < \varepsilon.$$

   But $\varepsilon > 0$ was arbitrary, so $U(f) - L(f) = 0$, which implies that $U(f) = L(f)$ and f is Riemann-integrable on $[a,b]$. ■