MAT 2125
Elementary Real Analysis

Exercises – Solutions – Q49-Q54

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49. Let $f : \mathbb{R} \to \mathbb{R}$, let J be a closed interval in $\mathbb{R}$ and let $c \in J$.

If f_2 is the restriction of f to J, show that if f has a limit at c then f_2 has a limit at c. Show the converse is not necessarily true.

Proof. Suppose $\lim_{x\to c} f(x) = L$ exists.

Then, $\forall\varepsilon > 0$, $\exists\delta_\varepsilon > 0$ s.t. $|f(x) - L| < \varepsilon$ whenever $0 < |x - c| < \delta_\varepsilon$.

But $f_2(x) = f(x)$ for all $x \in J \subseteq \mathbb{R}$.

Then, $\forall\varepsilon > 0$, $\exists\delta_\varepsilon > 0$ (exactly as above) s.t. $|f_2(x)-L| = |f(x)-L| < \varepsilon$ whenever $0 < |x - c| < \delta_\varepsilon$ and $x \in J$, and so $\lim_{x\to c} f_2(x) = L$.

Now consider $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & \text{if } x \in (-\infty, 0) \cup (1, \infty) \\ 1 & \text{if } x \in [0, 1] \end{cases},$$

with $J = [0, 1]$ and $f_2 = f|_J$. Then $\lim_{x\to 1} f_2(x) = 1$ but $\lim_{x\to 1} f(x)$ does not exist. ∎

50. Determine the following limits and state which theorems are used in each case.

(a) $\displaystyle\lim_{x\to 2}\sqrt{\frac{2x+1}{x+3}}$, $(x>0)$;

(b) $\displaystyle\lim_{x\to 2}\frac{x^2-4}{x-2}$, $(x>0)$;

(c) $\displaystyle\lim_{x\to 0}\sqrt{\frac{(x+1)^2-1}{x}}$, $(x>0)$, and

(d) $\displaystyle\lim_{x\to 1}\frac{\sqrt{x}-1}{x-1}$, $(x>0)$.

Proof. We will do (c) and just give the answers to the others.

Consider the sequence $(x_n) = (\frac{1}{n})$. Then $x_n \to 0$, $x_n \neq 0\ \forall n \in \mathbb{N}$, and

$$\frac{(x_n+1)^2-1}{x_n} = \frac{\left(\frac{1}{n}+1\right)^2-1}{\frac{1}{n}} = \frac{1}{n}+2 \to 2.$$

Hence, if $\lim_{x\to 0}\frac{(x+1)^2-1}{x}$ exists, its value must be 2, by theorem 26.

Let $\varepsilon > 0$. Set $\delta_\varepsilon = \varepsilon$. Then when $0 < |x-0| < \delta_\varepsilon$ and $x > 0$, we have

$$\left|\frac{(x+1)^2-1}{x} - 2\right| = \left|\frac{x^2+2x+1-1-2x}{x}\right| = \left|\frac{x^2}{x}\right| = |x| = |x-0| < \delta_\varepsilon = \varepsilon.$$

(a) 1 (b) 4 (d) $\frac{1}{2}$ ■

51. Give examples of functions f and g such that f and g do not have limits at point c, but both $f + g$ and fg have limits at c.

Proof. Let $f,g:\mathbb{R}\to\mathbb{R}$ be defined by

$$f(x)=\begin{cases}1 & x\geq 0\\ -1 & x<0\end{cases}$$

and $g(x)=-f(x)$ for all $x\in\mathbb{R}$.

Then $f(x)+g(x)\equiv 0$ and $f(x)g(x)\equiv -1$. As a result,

$$\lim_{x\to 0}(f+g)(x)=0 \quad \text{and} \lim_{x\to 0}(fg)(x)=-1,$$

but the limits of f and g don't exist at 0 (see problem **??**). ■

52. Determine whether the following limits exist in $\mathbb{R}$:

(a) $\displaystyle\lim_{x\to 0} \sin\left(\frac{1}{x^2}\right)$, with $x \neq 0$;

(b) $\displaystyle\lim_{x\to 0} x\sin\left(\frac{1}{x^2}\right)$, with $x \neq 0$;

(c) $\displaystyle\lim_{x\to 0} \operatorname{sgn}\sin\left(\frac{1}{x}\right)$, with $x \neq 0$, and

(d) $\displaystyle\lim_{x\to 0} \sqrt{x}\sin\left(\frac{1}{x^2}\right)$, with $x > 0$.

Proof.

(a) Let $(x_n) = (\frac{1}{\sqrt{n\pi}})$ and $(y_n) = (\sqrt{\frac{2}{(4n+1)\pi}})$ for all $n \in \mathbb{N}$.

Then $x_n, y_n \to 0$ and $x_n, y_n \neq 0$ for all $n \in \mathbb{N}$. But

$$\sin\left(\frac{1}{x_n^2}\right) = \sin(n\pi) = 0 \quad \text{and} \quad \sin\left(\frac{1}{y_n^2}\right) = \sin\left(\frac{(4n+1)\pi}{2}\right) = 1$$

for all $n \in \mathbb{N}$.

Then $\sin(1/x_n^2) \to 0$ and $\sin(1/y_n^2) \to 1$. As $0 \neq 1$, $\lim_{x\to 0} \sin\left(\frac{1}{x^2}\right)$ doesn't exist.

(b) Consider the sequence $(x_n) = (\frac{1}{\sqrt{n\pi}})$. Then $x_n \to 0$ and $x_n \neq 0$ for all $n \in \mathbb{N}$. Furthermore,

$$x_n \sin\left(\frac{1}{x_n^2}\right) = \frac{1}{\sqrt{n\pi}} \sin(n\pi) = \frac{1}{\sqrt{n\pi}} \cdot 0 \to 0.$$

As a result, if $\lim_{x\to 0} x \sin\left(\frac{1}{x^2}\right)$ exists, it must take the value 0.
Let $\varepsilon > 0$. Set $\delta_\varepsilon = \varepsilon$. Then

$$\left| x \sin\left(\frac{1}{x^2}\right) - 0 \right| = |x| \left| \sin\left(\frac{1}{x^2}\right) \right| \leq |x| = |x - 0| < \delta_\varepsilon = \varepsilon$$

whenever $0 < |x - 0| < \delta_\varepsilon$ and $x > 0$. Hence $\lim_{x\to 0} x \sin\left(\frac{1}{x^2}\right) = 0$.

(c) Let $(x_n) = \left(\frac{2}{(2n+1)\pi}\right)$. Then $x_n \to 0$, $x_n \neq 0$ for all $n \in \mathbb{N}$ and

$$\operatorname{sgn}\left(\sin\left(\frac{1}{x_n}\right)\right) = \operatorname{sgn}\left((-1)^n\right) = (-1)^n,$$

which does not converge. Hence $\lim\limits_{x\to 0} \operatorname{sgn}\left(\sin\left(\frac{1}{x}\right)\right)$ does not exist.

(d) $\lim\limits_{x\to 0} \sqrt{x}\sin\left(\frac{1}{x^2}\right) = 0$, with the same proof as (b), save for $\delta_\varepsilon = \varepsilon^2$. ■

53. Let $f : \mathbb{R} \to \mathbb{R}$ be s.t. $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. Assume $\lim_{x\to 0} f(x) = L$ exists. Prove that $L = 0$ and that f has a limit at every point $c \in \mathbb{R}$.

Proof. As f is additive, $f(2x) = f(x+x) = f(x) + f(x) = 2f(x)$, so

$$L = \lim_{y\to 0} f(y) = \lim_{2x\to 0} f(2x) = \lim_{x\to 0} f(2x) = \lim_{x\to 0} 2f(x) = 2\lim_{x\to 0} f(x) = 2L;$$

hence $L = 2L$ and $L = 0$, i.e. $\lim_{x\to 0} f(x) = 0$.

Now, let $c \in \mathbb{R}$. Then

$$\begin{aligned}\lim_{x\to c} f(x) &= \lim_{x\to c} (f(x-c) + f(c)) = \lim_{x\to c} f(x-c) + \lim_{x\to c} f(c) \\ &= \lim_{y\to 0} f(y) + f(c) = 0 + f(c) = f(c).\end{aligned}$$

As f is defined on all of $\mathbb{R}$, $f(c)$ exists for all $c \in \mathbb{R}$, and so $\lim_{x\to c} f(x) = f(c)$ exists for all $c \in \mathbb{R}$. ■