MAT 2125
Elementary Real Analysis

Exercises – Solutions – Q54-Q60

Winter 2021

P. Boily (uOttawa)

54. Let $K > 0$ and let $f : \mathbb{R} \to \mathbb{R}$ satisfy the condition

$$|f(x) - f(y)| \leq K|x - y|$$

for all $x, y \in \mathbb{R}$. Show that f is continuous on $\mathbb{R}$.

Proof. Let $c \in \mathbb{R}$ and $\varepsilon > 0$. Set $\delta_\varepsilon = \frac{\varepsilon}{K}$. Then

$$|f(x) - f(c)| \leq K|x - c| < K\delta_\varepsilon < K\frac{\varepsilon}{K} = \varepsilon$$

whenever $|x - c| < \delta_\varepsilon$.

55. Let $f : (0,1) \to \mathbb{R}$ be bounded and s.t. $\lim_{x \to 0} f(x)$ does not exist.

Show that there are two convergent sequences $(x_n), (y_n) \subseteq (0,1)$ with $x_n, y_n \to 0$ and $f(x_n) \to \xi, f(y_n) \to \zeta$, but $\xi \neq \zeta$.

Proof. For $n \in \mathbb{N}$, let $I_n = (0, 1/n)$ and set

$$s_n = \sup f(I_n) \quad \text{and} \quad t_n = \inf f(I_n).$$

These are well defined as $f(I_n)$ is bounded. By construction, (s_n) is decreasing and (t_n) is increasing. Since

$$s_1 \geq s_n = \sup f(I_n) \geq \inf f(I_n) = t_n \geq t_1,$$

(s_n) is bounded below by t_1 and (t_n) is bounded above by s_1. Hence $s_n \to s$ and $t_n \to t$ exist, by the Monotone Convergence theorem.

For $n \in \mathbb{N}$, let $x_n, y_n \in I_n$ be s.t.

$$|f(x_n) - s_n| < \frac{1}{n} \quad \text{and} \quad |f(y_n) - t_n| < \frac{1}{n}.$$

This can always be done as $s_n - \frac{1}{n}$ and $t_n + \frac{1}{n}$ are not the supremum and the infimum, respectively, of $f(I_n)$.

Then, $x_n, y_n \to 0$ and $x_n, y_n \neq 0$ for all $n \in \mathbb{N}$. Furthermore, $f(x_n) \to s$ and $f(y_n) \to t$, by the Squeeze Theorem; indeed, $s_n - \frac{1}{n} < f(x_n) \leq s_n$, $t_n \leq f(y_n) < t_n + \frac{1}{n}$, $s_n \to s$, and $t_n \to t$, and the statement follows.

Now, suppose that $s = t = \ell$. Then $s_n, t_n \to \ell$. Let $\varepsilon > 0$. $\exists N_1, N_2 \in \mathbb{N}$ s.t. $|s_n - \ell| < \varepsilon$ whenever $n > N_1$ and $|t_n - \ell| < \varepsilon$ whenever $n > N_2$.

Set $N_\varepsilon = \max\{N_1, N_2\}$. Then

$$\ell - \varepsilon < t_n \leq s_n < \ell - \varepsilon$$

whenever $n > N_\varepsilon$. Set $\delta_\varepsilon = \frac{1}{N_\varepsilon}$. Then

$$\ell - \varepsilon < t_{N_\varepsilon} = \inf f(I_{N_\varepsilon}) \leq f(x) \leq \sup f(I_{N_\varepsilon}) \leq s_{N_\varepsilon} < \ell + \varepsilon,$$

i.e. $|f(x)-\ell| < \varepsilon$ whenever $0 < |x-0| < \frac{1}{N_\varepsilon} = \delta_\varepsilon$. Hence $\lim_{x\to 0} f(x) = \ell$, which contradicts the hypothesis that the limit does not exist.

As a result, $s \neq t$, which completes the proof. $\blacksquare$

56. Let $f:\mathbb{R}\to\mathbb{R}$ be continuous on $\mathbb{R}$ and let $P=\{x\in\mathbb{R}:f(x)>0\}$. If $c\in P$, show that there exists a neighbourhood $V_\delta(c)\subseteq P$.

Proof. Let $c \in P$. Then $f(c) > 0$ and $\exists \varepsilon_0 > 0$ s.t. $f(c) - \varepsilon_0 > 0$.

By continuity of f, $\exists \delta_{\varepsilon_0}$ s.t. $|f(x) - f(c)| < \varepsilon_0$ whenever $|x - c| < \delta_{\varepsilon_0}$.

Thus, $0 < f(c) - \varepsilon_0 < f(x)$ for all $x \in V_{\delta_{\varepsilon_0}}$, i.e. $V_{\delta_{\varepsilon_0}} \subseteq P$. ■

57. Prove that if an additive function is continuous at some point $c \in \mathbb{R}$, it is continuous on $\mathbb{R}$.

Proof. In the light of a previous question on the topic, it is sufficient to show that if $\lim_{x\to c} f(x) = f(c)$ for some $c \in \mathbb{R}$, then $\lim_{x\to 0} f(x) = 0$.

Let f be continuous at c. Then

$$\begin{aligned} f(c) &= \lim_{x\to c} f(x) = \lim_{x\to c} \left(f(x-c) + f(c)\right) \\ &= \lim_{x\to c} f(x-c) + \lim_{x\to c} f(c) = \lim_{y\to 0} f(y) + f(c), \end{aligned}$$

hence $\lim_{y\to 0} f(y) = 0$, which completes the proof. ■

58. If f is a continuous additive function on $\mathbb{R}$, show that $f(x) = cx$ for all $x \in \mathbb{R}$, where $c = f(1)$.

Proof. Let $n \in \mathbb{N}$. Then

$$f(1) = f\left(\frac{n}{n}\right) = f\left(\frac{1}{n} + \cdots + \frac{1}{n}\right) = f\left(\frac{1}{n}\right) + \cdots + f\left(\frac{1}{n}\right) = nf\left(\frac{1}{n}\right),$$

hence $\frac{1}{n}f(1) = f\left(\frac{1}{n}\right)$.

Set $c = f(1)$. Let $y \in \mathbb{Q}$. Then $y = \frac{m}{n}$, where $m \in \mathbb{Z}$ and $n \in \mathbb{N}^{\times}$, and

$$f(y) = f\left(\frac{m}{n}\right) = mf\left(\frac{1}{n}\right) = m\frac{1}{n}f(1) = yc.$$

Let $x \in \mathbb{R}$. Since x is a limit point of $\mathbb{Q}$, $\exists (x_n) \subseteq \mathbb{Q}$ s.t. $x_n \to x$, with $x_n \neq x$ for all $n \in \mathbb{N}$. But $f(x_n) \to f(x)$, by continuity, so $f(x_n) = cx_n \to cx$, by the above discussion. Hence, $f(x) = cx$. ■

59. Let $I = [a, b]$ and $f : I \to \mathbb{R}$ be a continuous function on I s.t. $\forall x \in I$, $\exists y \in I$ s.t. $|f(y)| \leq \frac{1}{2}|f(x)|$. Show $\exists c \in I$ s.t. $f(c) = 0$.

Proof. Let $x_1 \in I$. By hypothesis, $\exists x_2 \in I$ s.t.

$$|f(x_2)| \leq \frac{1}{2}|f(x_1)|.$$

Since $x_2 \in I$, $\exists x_3 \in I$ s.t.

$$|f(x_3)| \leq \frac{1}{2}|f(x_2)| \leq \frac{1}{2}\left(\frac{1}{2}|f(x_1)|\right) = \frac{1}{2^2}|f(x_1)|,$$

and so on. The sequence $(x_n) \subseteq I$ thusly built satistfies

$$0 \leq |f(x_n)| \leq \frac{1}{2^{n-1}}|f(x_1)|,$$

by induction (can you show this?).

Then $\lim_{n\to\infty} |f(x_n)| = 0$, by the Squeeze Theorem, and so $f(x_n) \to 0$.

As (x_n) is bounded, it has a convergent subsequence (x_{n_k}) (by the Bolzano-Weierstrass Theorem) whose limit c is in I (because $a \leq x_n \leq b$ for all n).

Since $\big(f(x_{n_k})\big)$ is a subsequence of $(f(x_n))$, then

$$\lim_{k\to\infty} f(x_{n_k}) = 0.$$

However,

$$\lim_{k\to\infty} f(x_{n_k}) = f\left(\lim_{k\to\infty} x_{n_k}\right) = f(c),$$

as f is continuous. Hence $f(c) = 0$. ■

60. Show that every polynomial with odd degree has at least one real root.

Proof. Let

$$f(x) = a_{2n+1}x^{2n+1} + a_{2n}x^{2n} + \cdots + a_1x + a_0,$$

where $a_i \in \mathbb{R}$ for $i = 0, \ldots, 2n+1$. Assume that $a_{2n} \neq 0$ (if that is not the case, the proof will proceed in a similar fashion, but a_{2n} will be replaced by the first a_i that is non-zero, starting with a_{2n-1}; if all coefficients are 0, then the real root is 0).

Let

$$M = \max\left\{(2n+1)\frac{|a_{2n}|}{|a_{2n+1}|}, \left(\frac{|a_{2n-k}|}{|a_{2n}|}\right)^{1/k}; k = 1, \ldots, 2n\right\}.$$

Then, whenever $|x| \geq M$,

- $|a_{2n}||x^{2n}| \geq |a_{2n}||x^{2n}|$;
- $|a_{2n}||x^{2n}| \geq |a_{2n-1}||x^{2n-1}|$;
- $\cdots$;
- $|a_{2n}||x^{2n}| \geq |a_1||x|$, and
- $|a_{2n}||x^{2n}| \geq |a_0|$,

and so

$$\begin{aligned}|a_{2n}x^{2n} + \cdots a_0| \leq |a_{2n}||x^{2n}| + \cdots + |a_0| \leq |a_{2n}||x^{2n}| + \cdots + |a_{2n}||x^{2n}| \\ = (2n+1)|a_{2n}||x^{2n}| \leq |a_{2n+1}||x^{2n+1}| = |a_{2n+1}x^{2n+1}|.\end{aligned}$$

Then $f(M+1)f(-M-1) < 0$. As f is continuous on $[-M-1, M+1]$, $\exists c \in [-M-1, M+1]$ s.t. $f(c) = 0$, by the IVT. ■