MAT 2125
Elementary Real Analysis

Exercises – Solutions – Q62-Q65

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62. Show that $f(x) = \frac{1}{x^2}$ is uniformly continuous on $A = [1, \infty)$, but not on $B = (0, \infty)$.

Proof. If $x, y \in A$, then $x, y \geq 1$. In particular, $|x| = x$ and $|y| = y$, and $\frac{1}{x^2y}, \frac{1}{xy^2} \leq 1$.

Let $\varepsilon > 0$ and set $\delta_\varepsilon = \frac{\varepsilon}{2}$. Then

$$\begin{aligned}|f(x) - f(y)| &= \left|\frac{1}{x^2} - \frac{1}{y^2}\right| = \left|\frac{y^2 - x^2}{x^2y^2}\right| = \frac{|y+x||y-x|}{x^2y^2} \\ &= |y - x|\left(\frac{y}{x^2y^2} + \frac{x}{x^2y^2}\right) \\ &= |x - y|\left(\frac{1}{x^2y} + \frac{1}{xy^2}\right) \leq 2|x - y| < 2\delta_\varepsilon = \varepsilon\end{aligned}$$

whenever $|x - y| < \delta_\varepsilon$ and $x, y \in A$.

We show that the negation of the definition of uniform continuity holds on B.

Let $\varepsilon = 1$ and $\delta > 0$. Then, $\exists N \in \mathbb{N}$ s.t. $\frac{1}{N^2} < \delta$. Set $x_N = \frac{1}{N}$ and $y_N = \frac{1}{N+1}$. Clearly, $x_N, y_N \in B$ and

$$|x_N - y_N| = \left|\frac{1}{N} - \frac{1}{N+1}\right| = \frac{1}{N(N+1)} \leq \frac{1}{N^2} < \delta.$$

However,

$$|f(x_N) - f(y_N)| = |N^2 - (N+1)^2| = 2N + 1 > \varepsilon,$$

that is, f is not uniformly continuous on B. ■

63. If $f(x) = x$ and $g(x) = \sin x$, show that f and g are both uniformly continuous on $\mathbb{R}$ but that their product is not uniformly continuous on $\mathbb{R}$.

Proof. Let $\varepsilon > 0$ and set $\delta_\varepsilon = \varepsilon$. Then

$$|f(x) - f(y)| = |x - y| < \delta_\varepsilon = \varepsilon$$

and

$$\begin{aligned}|g(x) - g(y)| = |\sin x - \sin y| &= 2\left|\sin\left(\frac{1}{2}(x-y)\right)\cos\left(\frac{1}{2}(x+y)\right)\right| \\ &\leq 2\frac{1}{2}|x-y| \cdot 1 = |x-y| < \delta_\varepsilon = \varepsilon\end{aligned}$$

(the second-last inequality can be obtained using Taylor's theorem on $\sin$, see chapter 5), whenever $|x - y| < \delta_\varepsilon$ and $x, y \in \mathbb{R}$. Hence f and g are both uniformly continuous.

Set $h(x) = x \sin x$. Let $\varepsilon = 1$ and $\delta > 0$. Them $\exists N \in \mathbb{N}$ s.t. $\frac{1}{N} < \delta$ and $K \in \mathbb{N}$ s.t.

$$K > \frac{1}{4}\left(1 - \cos\frac{1}{N}\right)^{-1} + 3.$$

Define

$$x_K = \frac{4K-3}{2}\pi \quad \text{and} \quad y_K = \frac{4K-3}{2}\pi - \frac{1}{N}.$$

Then $|x_K - y_K| = \frac{1}{N} < \delta$ and

$$|h(x_K) - h(y_K)| \geq \frac{4K-3}{2}\pi\left(1 - \cos\frac{1}{N}\right) > \frac{\pi}{2} > 1 = \varepsilon,$$

and so h is not uniformly continuous. ■

64. Let $A \subseteq \mathbb{R}$ and suppose that f has the following property:

$\forall \varepsilon > 0$, $\exists g_\varepsilon : A \to \mathbb{R}$ s.t. g_ε is uniformly continuous on A with $|f(x) - g_\varepsilon(x)| < \varepsilon$ for all $x \in A$.

Show f is uniformly continuous on A.

Proof. Let $\varepsilon > 0$.

The $\frac{\varepsilon}{3} > 0$ and there exists $g_{\varepsilon/3}$ as in the hypothesis: hence $\exists \eta_{\varepsilon/3} > 0$ s.t. $|g_{\varepsilon/3}(x) - g_{\varepsilon/3}(y)| < \frac{\varepsilon}{3}$ whenever $|x - y| < \eta_{\varepsilon/3}$ and $x, y \in A$.

Set $\delta_\varepsilon = \eta_{\varepsilon/3}$. Then

$$\begin{aligned}|f(x) - f(y)| &= |f(x) - g_{\varepsilon/3}(x) + g_{\varepsilon/3}(x) - g_{\varepsilon/3}(y) + g_{\varepsilon/3}(y) - f(y)| \\ &\leq |f(x) - g_{\varepsilon/3}(x)| + |g_{\varepsilon/3}(x) - g_{\varepsilon/3}(y)| + |g_{\varepsilon/3}(y) - f(y)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon\end{aligned}$$

whenever $|x - y| < \delta_\varepsilon$ and $x, y \in A$.

Hence, f is uniformly continuous on A. ■

65. Prove that a continuous $p-$periodic fonction on $\mathbb{R}$ is bounded and uniformly continuous on $\mathbb{R}$.

Proof. Since f is continuous, then $|f|$ is also continuous, being the composition of two continuous functions.

As f is $p-$periodic, $\exists c \in [0,p]$ s.t.

$$\sup_{x\in\mathbb{R}} |f(x)| = \sup_{x\in[0,p]} |f(x)| = |f(c)|,$$

by the Max/Min Theorem. Hence f is bounded by $|f(c)|$ on $\mathbb{R}$.

Now, let $\varepsilon > 0$.

By hypothesis, f is continuous on the closed interval $[-1,p+1]$, which implies that that f is uniformly continuous on $[-1,p+1]$ (Theorem 38).

Then, $\exists \delta_\varepsilon > 0$ s.t. $|f(x) - f(y)| < \varepsilon$ whenever $|x - y| < \delta_\varepsilon$ and $x, y \in [-1, p+1]$.

Without loss of generality, we can assume that $\delta_\varepsilon < 1$ (why?). Let $x, y \in \mathbb{R}$ s.t. $|x - y| < \delta_\varepsilon$.

Then $\exists k \in \mathbb{Z}$ and $\alpha, \beta \in [-1, p+1]$ s.t. $x = \alpha + kp$ and $y = \beta + kp$.

Thus $|\alpha - \beta| = |x - y| < \delta_\varepsilon$ and $|f(x) - f(y)| = |f(\alpha) - f(\beta)| < \varepsilon$, since f is uniformly continuous on $[-1, p+1]$; as a result, f is uniformly continuous. ■