MAT 2125
Elementary Real Analysis

Exercises – Solutions – Q66-Q68

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66. Use the definition to find the derivative of the function defined by $g(x) = \frac{1}{x}$, $x \in \mathbb{R}$, $x \neq 0$.

Proof. From calculus, we "know" that $g'(x) = -\frac{1}{x^2}$.

Let $c \in \mathbb{R}$ s.t. $c \neq 0$. Set $a_c = \frac{c}{2}$ and $b_c = \frac{3c}{2}$. Clearly, if $c > 0$, $0 < a_c < c < b_c$, whereas $b_c < c < a_c < 0$ if $c < 0$. In both cases,

$$\frac{1}{|x|} \leq \frac{1}{|a_c|}$$

whenever x lies between a_c and b_c. We restrict g on the interval between a_c and b_c (denote this interval by A).

Let $\varepsilon > 0$ and set $\delta_\varepsilon = |a_c|c^2\varepsilon$. Then whenever $0 < |x - c| < \delta_\varepsilon$ and $x \in A$, we have

$$\left| \frac{\frac{1}{x} - \frac{1}{c}}{x - c} + \frac{1}{c^2} \right| = \left| \frac{c - x}{xc(x - c)} + \frac{1}{c^2} \right| = \left| \frac{1}{c^2} - \frac{1}{xc} \right| = \frac{|x - c|}{|x|c^2} \leq \frac{|x - c|}{|a_c|c^2} < \frac{\delta_\varepsilon}{|a_c|c^2} = \varepsilon.$$

67. Prove that the derivative of an even differentiable function is odd, and vice-versa.

Proof. If f is even, then $f(x) = f(-x)$ for all $x \in \mathbb{R}$. Let $g(x) = f(-x)$. Then g is differentiable by the chain rule and $f(x) = g(x)$ for all $x \in \mathbb{R}$.

Furthermore,

$$f'(x) = g'(x) = (f(-x))' = f'(-x) \cdot -1,$$

that is, $-f'(-x) = f'(-x)$, or f' is odd. The other statement is proved similarly. ■

68. Let $a > b > 0$ and $n \in \mathbb{N}$ with $n \geq 2$.

Show that $a^{1/n} - b^{1/n} < (a-b)^{1/n}$.

Proof. Consider the continuous function $f : [1,\infty) \to \mathbb{R}$ defined by $f(x) = x^{1/n} - (x-1)^{1/n}$, whose derivative is

$$f'(x) = \frac{1}{n}\left(x^{\frac{1-n}{n}} - (x-1)^{\frac{1-n}{n}}\right).$$

Now,

$$0 \leq x-1 < x, \quad \forall x \geq 1 \implies 0 \leq (x-1)^n < x^n, \quad \forall x \geq 1, n \geq 2$$
$$\therefore 0 \leq (x-1)^{\frac{n}{n-1}} < x^{\frac{n}{n-1}}, \quad \forall x \geq 1, n \geq 2$$

and so

$$\frac{1}{x^{\frac{n}{n-1}}} < \frac{1}{(x-1)^{\frac{n}{n-1}}},$$

or $x^{\frac{1-n}{n}} < (x-1)^{\frac{1-n}{n}}$ for all $x \geq 1$, $n \geq 2$.

Hence $f'(x) < 0$ for all $x \in [1, \infty)$, that is f is strictly decreasing over $[1, \infty)$. But $f(\frac{a}{b}) < f(1)$, as $\frac{a}{b} > 1$. But

$$f\left(\frac{a}{b}\right) = \left(\frac{a}{b}\right)^{\frac{1}{n}} - \left(\frac{a}{b} - 1\right)^{\frac{1}{n}} = \frac{1}{b^{\frac{1}{n}}}\left(a^{\frac{1}{n}} - (a-b)^{\frac{1}{n}}\right)$$

and $f(1) = 1$, so

$$\frac{1}{b^{\frac{1}{n}}}\left(a^{\frac{1}{n}} - (a-b)^{\frac{1}{n}}\right) < 1,$$

that is $a^{\frac{1}{n}} - (a-b)^{\frac{1}{n}} < b^{\frac{1}{n}}$, which completes the proof. ■