MAT 2125
Elementary Real Analysis

Exercises – Solutions – Q69-Q72

Winter 2021

P. Boily (uOttawa)

69. Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Show that if $\lim_{x \to a} f'(x) = A$, then $f'(a)$ exists and equals A.

Proof. Let $x \in (a,b)$. By the Mean Value Theorem, $\exists c_x \in (a,x)$ s.t.

$$\frac{f(x)-f(a)}{x-a} = f'(c_x).$$

When $x \to a$, $c_x \to a$ (indeed, let $\varepsilon > 0$ and set $\delta_\varepsilon = \varepsilon$; then $|c_x - a| < |x-a| < \delta_\varepsilon = \varepsilon$ whenever $0 < |x-a| < \delta_\varepsilon$). Then

$$\lim_{x\to a} f'(c_x) = \lim_{c_x \to a} f'(c_x) = A$$

by hypothesis. Hence $\lim\limits_{x\to a} f'(x)$ exists and so

$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim_{x\to a} f'(x) = A$$

exists. ■

70. If $x > 0$, show $1 + \frac{1}{2}x - \frac{1}{8}x^2 \leq \sqrt{1+x} \leq 1 + \frac{1}{2}x$.

Proof. Let $x_0 = 0$ and $f(x) = \sqrt{1+x}$. According to Taylor's theorem, since f is C^3 when $x > 0$, $f(x) = P_1(x) + R_1(x)$ and $f(x) = P_2(x) + R_2(x)$, where

$$P_1(x) = f(x_0) + f'(x_0)(x - x_0) = \sqrt{1+0} + \frac{1}{2\sqrt{1+0}}x = 1 + \frac{1}{2}x$$

$$P_2(x) = P_1(x) + \frac{f''(x_0)}{2}(x - x_0)^2 = 1 + \frac{1}{2}x - \frac{1}{8\sqrt[3]{1+0}}x^2 = 1 + \frac{1}{2}x - \frac{1}{8}x^2$$

$$R_1(x) = \frac{f''(c_1)}{2}(x - x_0)^2 = -\frac{1}{8\sqrt[3]{1+c_1}}x^2, \quad \text{for some } c_1 \in [0, x]$$

$$R_2(x) = \frac{f'''(c_2)}{6}(x - x_0)^3 = \frac{3}{48\sqrt[5]{1+c_2}}x^3, \quad \text{for some } c_2 \in [0, x].$$

When $x > 0$, $R_1(x) \leq 0$ and $R_2(x) \geq 0$, so $P_2(x) \leq f(x) \leq P_1(x)$. ■

71. Show directly that the function defined by $h(x) = x^2$ is Riemann-integrable over $[a, b]$, $b > a \geq 0$. Furthermore show that $\int_a^b h = \frac{b^3-a^3}{3}$.

Proof. Let $P_n = \left\{x_i = a + \frac{b-a}{n} \cdot i \mid i = 0, \ldots, n\right\}$ be a partition of $[a,b]$. Set $m_i = \inf\{h(x) \mid x \in [x_{i-1}, x_i]\}$, for $i = 1, \ldots, n$.

With this notation, we have

$$L(P_n; h) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}) = \frac{b-a}{n} \sum_{i=1}^{n} m_i.$$

But $h'(x) = 2x \geq 0$ when $x \geq 0$, and so h is increasing on $[a,b]$.

Consequently, for $i = 1, \ldots, n$, we have

$$m_i = x_{i-1}^2 = \left(a + \frac{b-a}{n}(i-1)\right)^2 = a^2 + 2\frac{a(b-a)}{n}(i-1) + \frac{(b-a)^2}{n^2}(i-1)^2.$$

The lower sum of h associated to P_n is thus

$$\begin{aligned} L(P_n;h) &= \frac{b-a}{n}\sum_{i=1}^{n}\left(a^2+2\frac{a(b-a)}{n}(i-1)+\frac{(b-a)^2}{n^2}(i-1)^2\right) \\ &= \frac{na^2(b-a)}{n}+\frac{2a(b-a)^2}{n^2}\sum_{i=1}^{n}(i-1)+\frac{(b-a)^3}{n^3}\sum_{i=1}^{n}(i-1)^2 \\ &= a^2(b-a)+\frac{2a(b-a)^2}{n^2}\cdot\frac{n(n-1)}{2}+\frac{(b-a)^3}{n^3}\cdot\frac{n(n-1)(2n-1)}{6} \\ &= a^2(b-a)+a(b-a)^2\left(1-\frac{1}{n}\right)+\frac{(b-a)^3}{6}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right). \end{aligned}$$

But for the lower sum of h on $[a,b]$, we have

$$\begin{aligned} L(h) &= \sup\{L(P;h) \mid P \in \mathcal{P}([a,b])\} \geq \sup_{n\in\mathbb{N}}\{L(P_n;h)\} \\ &= \sup_{n\in\mathbb{N}}\left\{a^2(b-a)+a(b-a)^2\left(1-\tfrac{1}{n}\right)+\tfrac{(b-a)^3}{6}\left(1-\tfrac{1}{n}\right)\left(2-\tfrac{1}{n}\right)\right\} \\ &= \lim_{n\to\infty}\left[a^2(b-a)+a(b-a)^2\left(1-\tfrac{1}{n}\right)+\tfrac{(b-a)^3}{6}\left(1-\tfrac{1}{n}\right)\left(2-\tfrac{1}{n}\right)\right] \\ &= a^2(b-a)+a(b-a)^2+\frac{(b-a)^3}{6}\cdot 2 = \frac{b^3-a^3}{3} \end{aligned}$$

Similarly, we can show that $U(h) \leq \frac{b^3-a^3}{3}$, from which we derive $\frac{b^3-a^3}{3} \leq L(h) \leq U(h) \leq \frac{b^3-a^3}{3}$ and so $L(h) = U(h) = \int_a^b h = \frac{b^3-a^3}{3}$. ■

72. Prove that $\int_0^1 g = \frac{1}{2}$ if

$$g(x) = \begin{cases} 1 & x \in (\frac{1}{2}, 1] \\ 0 & x \in [0, \frac{1}{2}] \end{cases}.$$

Is that still true if $g(\frac{1}{2}) = 7$ instead?

Proof. Let $\varepsilon > 0$ and define the associated partition

$$P_\varepsilon = \{0, \frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon, 1\}.$$

Since g is bounded on $[0,1]$, $L(g) \leq U(g)$ exist and

$$L(g) \geq L(P_\varepsilon; g) = \frac{1}{2} - \varepsilon \quad \text{and} \quad U(g) \leq U(P_\varepsilon; g) = \frac{1}{2} + \varepsilon.$$

Hence

$$\frac{1}{2} - \varepsilon \leq L(g) \leq U(g) \leq \frac{1}{2} + \varepsilon, \quad \text{for all } \varepsilon > 0.$$

Since $\varepsilon > 0$ is arbitrary, then $\frac{1}{2} \leq L(g) \leq U(g) \leq \frac{1}{2}$; by definition, g is Riemann integrable on $[0,1]$ and $L(g) = U(g) = \int_a^b f = \frac{1}{2}$.

If instead $g(1/2) = 7$, the exact same work as above yields

$$\frac{1}{2} - \varepsilon \leq L(g) \leq U(g) \leq \frac{1}{2} + 13\varepsilon, \quad \text{for all } \varepsilon > 0.$$

Since $\varepsilon > 0$ is arbitrary, then $\frac{1}{2} \leq L(g) \leq U(g) \leq \frac{1}{2}$; by definition, g is also Riemann integrable on $[0,1]$ and $L(g) = U(g) = \int_a^b f = \frac{1}{2}$. ■