MAT 2125
Elementary Real Analysis

Exercises – Solutions – Q73-Q75

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73. Let $f : [a, b] \to \mathbb{R}$ be bounded and s.t. $f(x) \geq 0$ for all $x \in [a, b]$.

Show $L(f) \geq 0$.

Proof. As f is bounded on $[a, b]$, $L(f)$ exists and the set

$$\{f(x) : x \in [a, b]\} \neq \varnothing$$

is bounded below.

By completeness of $\mathbb{R}$, $m_1 = \inf\{f(x) : x \in [a, b]\}$ exists.

Furthermore, $m_1 \geq 0$ since $f(x) \geq 0$ for all $x \in [a, b]$.

Let $P = \{x_0, x_1\} = \{a, b\}$ be the trivial partition of $[a, b]$.

Then $L(f) \geq L(P; f) = m_1(b - a) \geq 0$. ■

74. Let $f:[a,b]\to\mathbb{R}$ be increasing on $[a,b]$. If P_n partitions $[a,b]$ into n equal parts, show that

$$0 \leq U(P_n;f) - \int_a^b f \leq \frac{f(b)-f(a)}{n}(b-a).$$

Proof. As f is increasing, it is monotone and thus Riemann integrable by a result seen in class (Theorem 53).

Then $L(f) = U(f) = \int_a^b f$.

Let

$$P_n = \{x_i = a + i\tfrac{b-a}{n} : i = 0, \ldots, n\}$$

be the partition of $[a, b]$ into n equal sub-intervals.

By definition, $L(P_n; f) \leq \int_a^b f$ and $U(P_n; f) \geq \int_a^b f$. Then

$$U(P_n; f) - L(P_n; f) \geq U(P_n; f) - \int_a^b f \geq \int_a^b f - \int_a^b f = 0.$$

In particular, $U(P_n; f) - \int_a^b f \geq 0$. As f is increasing on $[a, b]$,

$$M_i = \sup_{[x_{i-1},x_i]}\{f(x)\} = f(x_i), \quad m_i = \inf_{[x_{i-1},x_i]}\{f(x)\} = f(x_{i-1}), \quad \text{and}$$

$$\begin{aligned} U(P_n;f) - L(P_n;f) &= \sum_{i=1}^{n}(M_i - m_i)(x_i - x_{i-1}) \\ &= \frac{b-a}{n}\sum_{i=1}^{n}(f(x_i) - f(x_{i-1})) = \frac{b-a}{n}(f(b) - f(a)). \end{aligned}$$

Since $L(P_n;f) \leq \int_a^b f$, then

$$\frac{b-a}{n}(f(b) - f(a)) = U(P_n;f) - L(P_n;f) \geq U(P_n;f) - \int_a^b f \geq 0. \quad \blacksquare$$

75. Let $f : [a, b] \to \mathbb{R}$ be an integrable function and let $\varepsilon > 0$.

    If P_ε is the partition whose existence is asserted by the Riemann Criterion, show that $U(P; f) - L(P; f) < \varepsilon$ for all refinement P of P_ε.

Proof. Let P be a refinement of P_ε.

Then $U(P_{\varepsilon;f}) \geq U(P;f)$ and $L(P_\varepsilon;f) \leq L(P;f)$, and so

$$U(P_\varepsilon;f) \geq U(P;f) \geq L(P;f) \geq L(P_\varepsilon;f).$$

By the Riemann Criterion, $U(P_\varepsilon;f) < \varepsilon + L(P_\varepsilon;f)$. Then

$$\varepsilon + L(P;f) \geq \varepsilon + L(P_\varepsilon;f) > U(P_\varepsilon;f) \geq U(P;f),$$

i.e. $\varepsilon + L(P;f) > U(P;f)$, which completes the proof. ■