MAT 2125
Elementary Real Analysis

Exercises – Solutions – Q76-Q79

Winter 2021

P. Boily (uOttawa)

76. Let $a > 0$ and $J = [-a, a]$. Let $f : J \to \mathbb{R}$ be bounded and let $\mathcal{P}^*$ be the set of all partitions P of J that contain 0 and are symmetric.

Show $L(f) = \sup\{L(P; f) : P \in \mathcal{P}^*\}$.

Proof. Let $\alpha = \sup\{L(P;f) : P \in \mathcal{P}^*\}$. By definition,

$$\alpha \leq L(f) = \sup\{L(P;f) : P \text{ is a partition of } [-a,a]\}.$$

Let $\varepsilon > 0$ and $P_\varepsilon = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[-a,a]$ s.t. $L(f) - \varepsilon < L(P_\varepsilon; f) \leq L(f)$. Such a partition exists as $L(f) - \varepsilon$ is not the supremum of the aforementioned set.

Consider the set

$$\{0, \pm x_0, \ldots, \pm x_n\}.$$

Eliminate all the repetitions from this set and re-order its elements in an increasing order. Denote the new set by Q_ε.

Then Q_ε is a refinement of P_ε and $Q_\varepsilon \in \mathcal{P}^*$; furthermore, $\alpha \geq L(Q_\varepsilon; f)$.

In particular,

$$L(f) - \varepsilon < L(P_\varepsilon; f) \leq L(Q_\varepsilon; f) \leq \alpha \leq L(f),$$

as $\varepsilon > 0$ is arbitrary, $L(f) = \alpha$. ■

77. Let J be as in the previous question and let f be integrable on J. If f is even (i.e. $f(-x) = f(x)$ for all x), show that

$$\int_{-a}^{a} f = 2\int_{0}^{a} f.$$

If f is odd (i.e. $f(-x) = -f(x)$ for all x), show that

$$\int_{-a}^{a} f = 0.$$

Proof. As f is integrable over $[-a,a]$, a result seen in class (Theorem 56) implies that f is integrable over $[0,a]$.

If f is even, let $P \in \mathcal{P}^*$. Then there is a partition $\tilde{P}$ of $[0,a]$ s.t. $L(P;f) = 2L(\tilde{P};f)$ and vice-versa.

Indeed, let

$$P = \{x_{-n},\ldots,x_{-1},x_0,x_1,\ldots,x_n\},$$

where $x_0 = 0$ and $x_{-i} = -x_i$ for all $i = 1,\ldots,n$.

Then $P \in \mathcal{P}^*$.

Let $m_i = \inf\{f(x) : x \in [x_{i-1},x_i]\}$, for $i = -n-1,\ldots,0,\ldots,n$.

Since f is even, $m_i = m_{-i+1}$ for $i = -n-1,\ldots,0,\ldots,n$.

Then

$$\begin{aligned}L(P;f) &= \sum_{i=-n-1}^{0} m_i(x_i - x_{i-1}) + \sum_{i=1}^{n} m_i(x_i - x_{i-1}) \\ &= 2\sum_{i=1}^{n} m_i(x_i - x_{i-1}) = L(\tilde{P};f),\end{aligned}$$

where $\tilde{P}$ is a partition of $[0,a]$. This, combined with the previous exercise, yields

$$\begin{aligned}\int_{-a}^{a} f &= \sup\{L(P;f) : P \in \mathcal{P}^*\} = \sup\{2L(\tilde{P};f) : \tilde{P} \text{ is a partition of } [0,a]\} \\ &= 2\sup\{L(\tilde{P};f) : \tilde{P} \text{ is a partition of } [0,a]\} = 2\int_{0}^{a} f.\end{aligned}$$

If f is odd, consider the function $h:\mathbb{R}\to\mathbb{R}$ given by

$$h(x)=\begin{cases}1 & \text{if } x\geq 0\\ -1 & \text{if } x<0\end{cases}.$$

The product fh is an even function, so

$$2\int_0^a f = 2\int_0^a hf = \int_{-a}^a hf = \int_{-a}^0 hf + \int_0^a hf = \int_{-a}^0 -f + \int_0^a f,$$

and so $\int_0^a f = \int_{-a}^0 -f = -\int_{-a}^0 f$. Then

$$\int_{-a}^a f = \int_{-a}^0 f + \int_0^a f = -\int_0^a f + \int_0^a f = 0. \quad \blacksquare$$

78. Give an example of a function $f : [0,1] \to \mathbb{R}$ that is not integrable on $[0,1]$, but s.t. $|f|$ is integrable on $[0,1]$.

Proof. Here is one example: $f:[0,1]\to\mathbb{R}$, defined by $f(x)=-1$ if $x\notin\mathbb{Q}$ and $f(x)=1$ if $x\in\mathbb{Q}$.

The proof that f is not Riemann integrable is similar to the one done in class.

How would you prove that $|f|\equiv 1$ is Riemann integrable? (Hint: what do $U(P;f)$ and $L(P;f)$ look like?)

79. Let $f : [a, b] \to \mathbb{R}$ be integrable on $[a, b]$. Show $|f|$ is integrable on $[a, b]$ directly (without using a result seen in class).

Proof. Let $\varepsilon > 0$.

By the Riemann Criterion, there exists a partition $P_\varepsilon = \{x_0, \ldots, x_n\}$ of $[a, b]$ s.t. $U(P_\varepsilon; f) - L(P_\varepsilon; f) < \varepsilon$.

For all $i = 1, \ldots, n$, let

$M_i = \sup\{f(x) : x \in [x_{i-1}, x_i]\}$ and $m_i = \inf\{f(x) : x \in [x_{i-1}, x_i]\}$.

For all $i = 1, \ldots, n$, then,

$$|f(x) - f(y)| \leq M_i - m_i \quad \text{on } [x_{i-1}, x_i].$$

As $||f(x)| - |f(y)|| \leq |f(x) - f(y)| \leq M_i - m_i$ for all $x, y \in [x_{i-1}, x_i]$, we have

$$\tilde{M}_i - \tilde{m}_i \leq M_i - m_i,$$

where

$$\tilde{M}_i = \sup\{|f(x)| : x \in [x_{i-1}, x_i]\} \quad \text{and} \quad \tilde{m}_i = \inf\{|f(x)| : x \in [x_{i-1}, x_i]\}$$

for all $i = 1, \ldots, n$. Then

$$\begin{aligned} U(P_\varepsilon; |f|) - L(P_\varepsilon; |f|) &= \sum_{i=1}^{n} \left(\tilde{M}_i - \tilde{m}_i\right)(x_i = x_{i-1}) \\ &\leq \sum_{i=1}^{n} (M_i - m_i)(x_i = x_{i-1}) = U(P_\varepsilon; |f|) - L(P_\varepsilon; |f|) < \varepsilon. \end{aligned}$$

By the Riemann Criterion, this shows that $|f|$ is integrable on $[a, b]$. ■