MAT 2125
Elementary Real Analysis

Exercises – Solutions – Q80-Q83

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80. If f is integrable on $[a, b]$ and

$$0 \le m \le f(x) \le M$$

for all $x \in [a, b]$, show that

$$m \le \left[\frac{1}{b-a}\int_a^b f^2\right]^{1/2} \le M.$$

Proof. By hypothesis, $m^2 \leq f^2(x) \leq M^2$ for all $x \in [a,b]$.

As f is integrable on $[a,b]$, so is f^2, by a result seen in class (Theorem 58).

Then

$$\int_a^b m^2 \leq \int_a^b f^2 \leq \int_a^b M^2$$

by the “Squeeze Theorem for Integrals” and so

$$m^2(b-a) \leq \int_a^b f^2 \leq M^2(b-a).$$

We obtain the desired result by re-arranging the terms and extracting square roots. ■

81. If f is continuous on $[a,b]$ and $f(x) \geq 0$ for all $x \in [a,b]$, show there exists $c \in [a,b]$ s.t.

$$f(c) = \left[\frac{1}{b-a}\int_a^b f^2\right]^{1/2}.$$

Proof. By the Max/Min theorem, $\exists x_0, x_1 \in [a,b]$ s.t.

$$m = \inf_{[a,b]}\{f(x)\} = f(x_0),\ M = \sup_{[a,b]}\{f(x)\} = f(x_1).$$

Then by the preceding exercise, we have

$$f(x_0) \leq \left[\frac{1}{b-a}\int_a^b f^2\right]^{1/2} \leq f(x_1).$$

As f is continuous on $[x_0, x_1]$ (or $[x_1, x_0]$), the IVT states $\exists c \in [a,b]$ s.t.

$$f(c) = \left[\frac{1}{b-a}\int_a^b f^2\right]^{1/2}. \qquad \blacksquare$$

82. If f is continuous on $[a, b]$ and $f(x) > 0$ for all $x \in [a, b]$, show that $\frac{1}{f}$ is integrable on $[a, b]$.

Proof. Since f is continous on $[a, b]$ it is integrable on $[a, b]$.

Since f is continuous and $[a, b]$ is a closed bounded interval, then $f([a, b]) = [m, M]$ is also closed bounded interval (a result seen in class).

Furthermore, $0 < m \leq M$ since $f(x) > 0$ for all $x \in [a, b]$.

Let $\varphi : [m, M] \to \mathbb{R}$ be defined by $\varphi(t) = \frac{1}{t}$. Then φ is continuous and bounded on $[m, M]$ and so $\varphi \circ f : [a, b] \to \mathbb{R}$, defined by $\varphi(f(x)) = \frac{1}{f(x)}$ is integrable on $[a, b]$, by the Composition Theorem 57. ■

83. Let f be continuous on $[a, b]$. Define $H : [a, b] \to \mathbb{R}$ by

$$H(x) = \int_x^b f \quad \text{for all } x \in [a, b].$$

Find $H'(x)$ for all $x \in [a, b]$.

Proof. Define $F(x) = \int_a^x f$.

Since f is continuous, F is differentiable and $F'(x) = f(x)$ for all $x \in [a, b]$, by the Fundamental Theorem of Calculus.

Then, by the Additivity Theorem,

$$F(x) + H(x) = \int_a^x f + \int_b^x f = \int_a^b f.$$

In particular,

$$H(x) = \int_a^b f - F(x).$$

As F is differentiable, $\int_a^b f - F(x)$ is also differentiable; so is H since $H'(x) = 0 - F'(x) = -f(x)$. ■