MAT 2125
Elementary Real Analysis

Notes

Winter 2021

P. Boily (uOttawa)

Theorem 1. (Archimedean Property)
Let $x \in \mathbb{R}$. *Then* $\exists n_x \in \mathbb{N}^\times$ *such that* $x < n_x$.

Proof. Suppose that there is no such integer. Then $x \geq n \ \forall n \in \mathbb{N}$.

Consequently, x is an upper bound of $\mathbb{N}^\times$. But $\mathbb{N}^\times$ is a non-empty subset of $\mathbb{R}$. Since $\mathbb{R}$ is complete, $\alpha = \sup \mathbb{N}^\times$ exists.

By definition of the supremum (the smallest upper bound), $\alpha - 1$ is not an upper bound of $\mathbb{N}^\times$ (otherwise α would not be the smallest upper bound, as $\alpha - 1 < \alpha$ would be a smaller upper bound).

Since $\alpha - 1$ is not an upper bound of $\mathbb{N}^\times$, $\exists m \in \mathbb{N}^\times$ such that $\alpha - 1 < m$. Using the properties of $\mathbb{R}$, we must then have $\alpha < m + 1 \in \mathbb{N}^\times$; that is, α is not an upper bound of $\mathbb{N}^\times$.

This contradicts the fact that $\alpha = \sup \mathbb{N}^\times$, and so, since $\mathbb{N}^\times \neq \varnothing$, x cannot be an upper bound of $\mathbb{N}^\times$. Thus $\exists n_x \in \mathbb{N}^\times$ such that $x < n_x$. ■

Theorem 2. (ARCHIMEDEAN PROPERTY; VARIANTS)
Let $x, y \in \mathbb{R}^+$. *Then* $\exists n_1, n_2, n_3 \geq 1$ *such that*

1. $x < n_1 y$;

2. $0 < \frac{1}{n_2} < y$, *and*

3. $n_3 - 1 \leq x < n_3$.

Proof.

1. Let $z = \frac{x}{y} > 0$. By the Archimedean property, $\exists n_1 \geq 1$ such that $z = \frac{x}{y} < n_1$. Then $x < n_1 y$.

2. If $x = 1$, then part 1 implies $\exists n_2 \geq 1$ such that $0 < 1 < n_2 y$. Then $0 < \frac{1}{n_2} < y$.

3. Let $L = \{m \in \mathbb{N}^\times : x < m\}$. By the Archimedean property, $L \neq \varnothing$. Indeed, there is at least one $n \geq 1$ such that $x < n$. By the well-ordering principle, L has a smallest element, say $m = n_3$. Then $n_3 - 1 \notin L$ (otherwise, $n_3 - 1$ would be the least element of L, which it is not) and so $n_3 - 1 \leq x < n_3$.

There are other variants, but these are the ones we'll use the most. ■

Theorem 3. (Bernoulli's Inequality)
Let $x \geq -1$. *Then* $(1+x)^n \geq 1+nx$, $\forall n \in \mathbb{N}$.

Proof. We prove the result by induction on n.

- If $n = 1$, then $(1+x)^1 = 1 + x \geq 1 + 1x$.
- Suppose that the result is true for $n = k$, that is $(1+x)^k \geq 1 + kx$. We have to show that it is also true for $n = k+1$. But

$$\begin{aligned}(1+x)^{k+1} &= (1+x)^k(1+x) \\ \boxed{\text{Ind. Hyp.}} &\geq (1+kx)(1+x) \\ &= 1 + (k+1)x + kx^2 \\ &\geq 1 + (k+1)x. \qquad \blacksquare\end{aligned}$$

(Where does the hypothesis $x \geq -1$ come in to play?)

Theorem 4. (Cauchy's Inequality)
If $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ are real numbers, then

$$\left(\sum a_i b_i\right)^2 \leq \left(\sum a_i^2\right)\left(\sum b_i^2\right).$$

(The indices are understood to run from 1 to n in what follows.) Furthermore, if $b_j \neq 0$ for one of $1 \leq j \leq n$, then equality holds if and only if $\exists s \in \mathbb{R}$ such that $a_i = sb_i$ for all $i = 1, \ldots, n$.

Proof. For any $t \in \mathbb{R}$,

$$0 \leq \sum (a_i + tb_i)^2 = \sum a_i^2 + 2t \sum a_i b_i + t^2 \sum b_i^2.$$

The right-hand side of this inequality is a polynomial of degree 2 in t.

It is always greater than or equal to 0: it has at most 1 real root, i.e. its discriminant

$$\left(2 \sum a_i b_i\right)^2 - 4 \left(\sum a_i^2\right) \left(\sum b_i^2\right) \leq 0,$$

and so

$$\left(\sum a_i b_i\right)^2 \leq \left(\sum a_i^2\right) \left(\sum b_i^2\right).$$

If all the b_i are 0, the equality holds trivially, as both the left and right side of the Cauchy inequality are 0.

So suppose $b_i \neq 0$ for at least one of the values j between 1 and n. We have two statements to prove. If $a_i = sb_i$ for all $i = 1, \ldots, n$ and $s \in \mathbb{R}$ is fixed then

$$\begin{aligned}\left(\sum a_i b_i\right)^2 &= \left(\sum s b_i^2\right)^2 = s^2\left(\sum b_i^2\right)^2 = s^2\left(\sum b_i^2\right)\left(\sum b_i^2\right) \\ &= \left(\sum s^2 b_i^2\right)\left(\sum b_i^2\right) = \left(\sum a_i^2\right)\left(\sum b_i^2\right).\end{aligned}$$

On the other hand, if

$$\left(\sum a_i b_i\right)^2 = \left(\sum a_i^2\right)\left(\sum b_i^2\right)$$

then

$$4\left(\sum a_i b_i\right)^2 - 4\left(\sum a_i^2\right)\left(\sum b_i^2\right) = 0.$$

But the left-hand side of this expression is the discriminant of the following polynomial of degree 2 in t:

$$\sum (a_i + tb_i)^2 = \sum a_i^2 + 2t \sum a_i b_i + t^2 \sum b_i^2.$$

Since the discriminant is 0, the polynomial has a unique root, say $t = -s$,

$$\therefore \sum (a_i - sb_i)^2 = 0.$$

Since $(a_i - sb_i)^2 \geq 0$ for all $i = 1, \ldots, n$, then

$$\begin{aligned} (a_i - sb_i)^2 &= 0 \quad \text{for all } i = 1, \ldots, n \\ \therefore a_i - sb_i &= 0 \quad \text{for all } i = 1, \ldots, n \\ \therefore a_i &= sb_i \quad \text{for all } i = 1, \ldots, n. \qquad \blacksquare \end{aligned}$$

Theorem 5. (Triangle Inequality)
If $a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{R}$*, then*

$$\left(\sum (a_i + b_i)^2\right)^{1/2} \leq \left(\sum a_i^2\right)^{1/2} + \left(\sum b_i^2\right)^{1/2}.$$

Furthermore, if $b_j \neq 0$ *for one of* $1 \leq j \leq n$*, then equality holds if and only if* $\exists s \in \mathbb{R}$ *such that* $a_i = sb_i$ *for all* $i = 1, \ldots, n$*.*

Proof. As

$$\begin{aligned}\sum(a_i+b_i)^2 &= \sum a_i^2 + 2\sum a_i b_i + \sum b_i^2 \\ \boxed{\text{Cauchy Ineq.}}\ &\leq \sum a_i^2 + 2\left(\sum a_i^2\right)^{1/2}\left(\sum b_i^2\right)^{1/2} + \sum b_i^2 \\ &= \left(\left(\sum a_i^2\right)^{1/2} + \left(\sum b_i^2\right)^{1/2}\right)^2.\end{aligned}$$

Taking the square root on both sides yields the desired result.

If all the b_i are 0, the equality holds trivially, as both the left and right side of the Triangle Inequality are $\left(\sum a_i^2\right)^{1/2}$.

So suppose $b_i \neq 0$ for at least one of the values j between 1 and n. We have two statements to prove.

If $a_i = sb_i$ for all $i = 1, \ldots, n$ and $s \in \mathbb{R}$ is fixed then

$$\left(\sum (a_i + b_i)^2\right)^{1/2} = \left(\sum (sb_i + b_i)^2\right)^{1/2} = \left(\sum (s+1)^2 b_i^2\right)^{1/2}$$
$$= \left((s+1)^2 \sum b_i^2\right)^{1/2} = (s+1)\left(\sum b_i^2\right)^{1/2}$$

and

$$\left(\sum a_i^2\right)^{1/2} + \left(\sum b_i^2\right)^{1/2} = \left(\sum s^2 b_i^2\right)^{1/2} + \left(\sum b_i^2\right)^{1/2}$$
$$= s\left(\sum b_i^2\right)^{1/2} + \left(\sum b_i^2\right)^{1/2} = (s+1)\left(\sum b_i^2\right)^{1/2}$$

and so equality holds.

On the other hand, if

$$\left(\sum (a_i+b_i)^2\right)^{1/2} = \left(\sum a_i^2\right)^{1/2} + \left(\sum b_i^2\right)^{1/2}$$

then

$$\sum (a_i+b_i)^2 = \left(\left(\sum a_i^2\right)^{1/2} + \left(\sum b_i^2\right)^{1/2}\right)^2.$$

Developing both sides of this expression yields

$$\sum a_i^2 + 2\sum a_i b_i + \sum b_i^2 = \sum a_i^2 + 2\left(\sum a_i^2\right)^{1/2}\left(\sum b_i^2\right)^{1/2} + \sum b_i^2,$$

or simply

$$\sum a_i b_i = \left(\sum a_i^2\right)^{1/2}\left(\sum b_i^2\right)^{1/2}.$$

Elevating both sides to the second power yields

$$\left(\sum a_i b_i\right)^2 = \left(\sum a_i^2\right)\left(\sum b_i^2\right).$$

By Theorem 4, $\exists s \in \mathbb{R}$ such that $a_i = s b_i$ for all $i = 1, \ldots, n$. $\blacksquare$

Theorem 6. (Absolute Value; Properties)
If $x, y \in \mathbb{R}$ *and* $\varepsilon > 0$, *then*

1. $|x| = \sqrt{x^2}$
2. $-|x| \leq x \leq |x|$
3. $|xy| = |x||y|$
4. $|x + y| \leq |x| + |y|$
5. $|x - y| \leq |x| + |y|$
6. $||x| - |y|| \leq |x - y|$
7. $|x - y| < \varepsilon \iff y - \varepsilon < x < y + \varepsilon$

Theorem 7. (Density of $\mathbb{Q}$)
Let $x, y \in \mathbb{R}$ *such that* $x < y$. *Then,* $\exists r \in \mathbb{Q}$ *such that* $x < r < y$.

Proof. There are three distinct cases.

1. If $x < 0 < y$, then select $r = 0$.

2. If $0 \leq x < y$, then $y - x > 0$ and $\frac{1}{y-x} > 0$.

   By the Archimedean property, $\exists n \geq 1$ such that

   $$n > \frac{1}{y-x} > 0.$$

   By that same property, $\exists m \geq 1$ such that $m - 1 \leq nx < m$. Since $n(y-x) > 1$, then $ny - 1 > nx$ and $nx \geq m - 1$.

   By transitivity of $<$, $ny - 1 > m - 1$, that is $ny > m$. But $m > nx$, so $ny > m > nx$ and $y > \frac{m}{n} > x$. Select $r = \frac{m}{n}$.

3. If $x < y \leq 0$, then $y-x > 0$ and $\frac{1}{y-x} > 0$. By the Archimedean property, $\exists n \geq 1$ such that
$$n > \frac{1}{y-x} > 0.$$
Note that $-nx > 0$. By yet another variant of that property (that we haven't explicitly stated in class, but it's not too much work to show it), $\exists m \geq 0$ such that $m < -nx \leq m+1$ or $-m-1 \leq nx < -m$.

Since $n(y-x) > 1$, then $ny - 1 > nx$ and $nx \geq -m-1$.

By transitivity of $<$, $ny-1 > -m-1$, that is $ny > -m$. But $-m > nx$, so $ny > -m > nx$ and $y > -\frac{m}{n} > x$. Select $r = -\frac{m}{n}$. ∎

Theorem 8. *If S is an infinite subset of a countable set A, then S is countable.*

Proof. https://youtu.be/MufHda7srwo

Theorem 9. *The set $\mathbb{Q}$ is countable.*

Proof. https://youtu.be/MufHda7srwo

Theorem 10. *The set $\mathbb{R}$ is uncountable.*

Proof. https://youtu.be/IJgtq4_JYQE

Theorem 11. (NESTED INTERVALS) *For every integer* $n \in \mathbb{N}$, *let* $[a_n, b_n] = I_n$ *be such tht*

$$I_1 \supseteq I_2 \supseteq \cdots I_n \supseteq I_{n+1} \supseteq \cdots$$

Then there exists $\psi, \eta \in \mathbb{R}$ *such that* $\psi \leq \eta$ *and* $\bigcup_{n\in\mathbb{N}} I_n = [\psi, \eta]$. *Furthermore, if* $\inf\{b_n - a_n \mid n \in \mathbb{N}\} = 0$, *then* $\psi = \eta$.

Proof. https://youtu.be/D6zHW5L_9L0

Theorem 12. (Unique Limit) *A convergent sequence (x_n) of real numbers has exactly one limit.*

Proof. Suppose that $x_n \to x'$ and $x_n \to x''$.

Let $\varepsilon > 0$. Then there exist 2 integers $N'_\varepsilon, N''_\varepsilon \in \mathbb{N}$ such that

$$|x_n - x'| < \varepsilon \text{ whenever } n > N'_\varepsilon \quad \text{and} \quad |x_n - x''| < \varepsilon \text{ whenever } n > N''_\varepsilon.$$

Set $N_\varepsilon = \max\{N'_\varepsilon, N''_\varepsilon\}$. Then whenever $n > N_\varepsilon$, we have

$$0 \leq |x' - x''| = |x' - x_n + x_n - x''| \leq |x_n - x'| + |x_n - x''| < \varepsilon + \varepsilon = 2\varepsilon.$$

Thus $0 \leq \frac{|x'-x''|}{2} < \varepsilon$.

But $\varepsilon > 0$ was arbitrary, so $\frac{|x'-x''|}{2} = 0 \implies x' = x''$. ■

Theorem 13. *Any convergent sequence (x_n) of real numbers is bounded.*

Proof. Let $(x_n) \subseteq \mathbb{R}$ converge to $x \in \mathbb{R}$. Then for $\varepsilon = 1$, say, $\exists N \in \mathbb{N}$ such that

$$|x_n - x| < 1 \quad \text{when } n > N.$$

Thanks to the reverse triangle inequality, we also have

$$|x_n| - |x| \leq |x_n - x| < 1 \quad \text{when } n > N,$$

so that $|x_n| < |x| + 1$ when $n > N$.

Now, set $M = \max\{|x_1|, \ldots, |x_N|, |x| + 1\}$. Then $|x_n| \leq M$ for all n and so (x_n) is bounded. ■

Theorem 14. (OPERATIONS ON CONVERGENT SEQUENCES)
Let $(x_n), (y_n)$ be convergent sequences, with $x_n \to x$ and $y_n \to y$. Let $c \in \mathbb{R}$. Then

1. $|x_n| \to |x|$;
2. $(x_n + y_n) \to (x + y)$;
3. $x_n y_n \to xy$ *and* $cx_n \to cx$;
4. $\frac{x_n}{y_n} \to \frac{x}{y}$, *if* $y_n, y \neq 0$ *for all* n.

Proof. We show each part using the definition of the limit of a sequence.

1. Let $\varepsilon > 0$. As $x_n \to x$, $\exists N'_\varepsilon$ such that $|x_n - x| < \varepsilon$ whenever $n > N'_\varepsilon$. But $||x_n| - |x|| \leq |x_n - x|$, according to theorem 6. Hence, for $\varepsilon > 0$, $\exists N_\varepsilon = N'_\varepsilon$ such that

$$||x_n| - |x|| \leq |x_n - x| < \varepsilon$$

whenever $n > N_\varepsilon$, i.e. $|x_n| \to |x|$.

2. Let $\varepsilon > 0$. Then $\frac{\varepsilon}{2} > 0$. As $x_n \to x$ and $y_n \to y$, $\exists N^x_{\frac{\varepsilon}{2}}, N^y_{\frac{\varepsilon}{2}}$ such that

$$|x_n - x| < \frac{\varepsilon}{2} \quad \text{and} \quad |y_n - y| < \frac{\varepsilon}{2} \tag{1}$$

whenever $n > N^x_{\frac{\varepsilon}{2}}$ and $n > N^y_{\frac{\varepsilon}{2}}$ respectively. Set $N_\varepsilon = \max\left\{N^x_{\frac{\varepsilon}{2}}, N^y_{\frac{\varepsilon}{2}}\right\}$.

Then, whenever $n > N_\varepsilon$ (so whenever n is strictly larger than $N^x_{\varepsilon/2}$ and $N^y_{\varepsilon/2}$ *at the same time*),

$$\begin{aligned} |(x_n + y_n) - (x + y)| = |(x_n - x) + (y_n - y)| &\leq |x_n - x| + |y_n - y| \\ \boxed{\text{by (1)}} &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \end{aligned}$$

i.e. $(x_n + y_n) \to (x + y)$.

3. According to theorem 13, (x_n) and (y_n) are bounded since they are convergent sequences. Then $\exists M_x, M_y \in \mathbb{N}$ such that

$$|x_n| < M_x \quad \text{and} \quad |y_n| < M_y$$

for all n.

Let $\varepsilon > 0$. Then $\frac{\varepsilon}{2M_x}, \frac{\varepsilon}{2M_y} > 0$. As $x_n \to x$, $y_n \to y$, $\exists N^x_{\frac{\varepsilon}{2M_y}}, N^y_{\frac{\varepsilon}{2M_x}} \in \mathbb{N}$ such that

$$|x_n - x| < \frac{\varepsilon}{2M_y} \quad \text{and} \quad |y_n - y| < \frac{\varepsilon}{2M_x} \tag{2}$$

whenever $n > N^x_{\frac{\varepsilon}{2M_y}}$ and $n > N^y_{\frac{\varepsilon}{2M_x}}$ respectively. Moreover, $|y| \leq M_y$ (otherwise $\frac{|y|-M_y}{2} > 0$. Then, for $\varepsilon = \frac{|y|-M_y}{2}$, we get

$$|y_n - y| \geq ||y| - |y_n|| \geq |y| - M_y = 2\varepsilon > \varepsilon$$

for all $n \in \mathbb{N}$, which contradicts the definition of $y_n \to y$).

Set $N_\varepsilon = \max\left\{N^x_{\frac{\varepsilon}{2M_x}}, N^y_{\frac{\varepsilon}{2M_y}}\right\}$. Then, whenever $n > N_\varepsilon$,

$$\begin{aligned}
|x_ny_n - xy| = |x_ny_n - x_ny + x_ny - xy| &= |x_n(y_n - y) + y(x_n - x)| \\
&\leq |x_n||y_n - y| + |y||x_n - x| \\
&< M_x|y_n - y| + M_y|x_n - x| \\
\boxed{\text{by (2)}} \quad &< M_x\frac{\varepsilon}{2M_x} + M_y\frac{\varepsilon}{2M_y} \\
&= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,
\end{aligned}$$

i.e. $x_ny_n \to xy$.

Furthermore, if the sequence (y_n) is given by $y_n = c$ for all n, then the preceding result yields $cx_n \to cx$, since $y_n = c \to c$ (You should show this).

4. It is enough to show $\frac{1}{y_n} \to \frac{1}{y}$ under the hypotheses above; then the result will hold by part 3. Since $y \neq 0$, $\frac{|y|}{2} > 0$. Hence, as $y_n \to y$, $\exists N_{|y|/2} \in \mathbb{N}$ such that $|y_n - y| < \frac{|y|}{2}$, whenever $n > N_{\frac{|y|}{2}}$. According to theorem 6,

$$|y| - |y_n| < |y - y_n| < \frac{|y|}{2}, \quad \text{and so}$$

$$\frac{|y|}{2} < |y_n| \quad \text{or} \quad \frac{1}{|y_n|} < \frac{2}{|y|} \tag{3}$$

whenever $n > N_{|y|/2}$ (these expressions make sense as neither y_n nor y is 0 for all n).

Let $\varepsilon > 0$. Then $|y|^2\frac{\varepsilon}{2} > 0$. As $y_n \to y$, $\exists N_{|y|^2\frac{\varepsilon}{2}} \in \mathbb{N}$ such that

$$|y_n - y| < |y|^2\frac{\varepsilon}{2} \tag{4}$$

whenever $n > N_{|y|^2\frac{\varepsilon}{2}}$. Set $N_\varepsilon = \max\left\{N_{\frac{|y|}{2}}, N_{|y|^2\frac{\varepsilon}{2}}\right\}$. Then, whenever $n > N_\varepsilon$,

$$\begin{aligned}\left|\frac{1}{y_n} - \frac{1}{y}\right| = \left|\frac{y - y_n}{y_n y}\right| &= \frac{|y - y_n|}{|y_n y|} \\ \boxed{\text{by (3)}} &< \frac{2|y - y_n|}{|y|^2} \\ \boxed{\text{by (4)}} &< \frac{2}{|y|^2} \cdot |y|^2\frac{\varepsilon}{2} = \varepsilon, \quad \text{i.e. } \frac{1}{y_n} \to \frac{1}{y}. \qquad \blacksquare\end{aligned}$$

Theorem 15. (Comparison Theorem for Sequences)
Let $(x_n), (y_n)$ be convergent sequences of real numbers with $x_n \to x$, $y_n \to y$, and $x_n \leq y_n$ $\forall n \in \mathbb{N}$. Then $x \leq y$.

Proof. Suppose that it is not the case, namely, that $x > y$. Then $x - y > 0$.

Set $\varepsilon = \frac{x-y}{2} > 0$. Since $x_n \to x$ and $y_n \to y$, $\exists N_\varepsilon^x, N_\varepsilon^y \in \mathbb{N}$ s.t.

$$|x_n - x| < \varepsilon \text{ whenever } n > N_\varepsilon^x \quad \text{and} \quad |y_n - y| < \varepsilon \text{ whenever } n > N_\varepsilon^y.$$

Let $N_\varepsilon = \max\{N_\varepsilon^x, N_\varepsilon^y\}$. Then, if $n > N_\varepsilon$, we have

$$y_n < y + \varepsilon = y + \frac{x-y}{2} = \frac{x+y}{2} = x - \frac{x-y}{2} = x - \varepsilon < x_n.$$

But this contradicts the assumption that $x_n \leq y_n$ for all n.

Consequently, $x \leq y$. ∎

Theorem 16. (Squeeze Theorem for Sequences)
Let $(x_n), (y_n), (z_n) \subseteq \mathbb{R}$ *be such that* $x_n, z_n \to \alpha$ *and* $x_n \leq y_n \leq z_n$*,* $\forall n \in \mathbb{N}$*. Then* $y_n \to \alpha$*.*

Proof. Let $\varepsilon > 0$. By convergence of $(x_n), (z_n)$ to α, $\exists N_\varepsilon^x, N_\varepsilon^z \in \mathbb{N}$ s.t.

$$|x_n - \alpha| < \varepsilon \text{ whenever } n > N_\varepsilon^x \quad \text{and} \quad |z_n - \alpha| < \varepsilon \text{ whenever } n > N_\varepsilon^z.$$

Let $N_\varepsilon = \max\{N_\varepsilon^x, N_\varepsilon^z\}$. Then, if $n > N_\varepsilon$, we have

$$\alpha - \varepsilon < x_n \leq y_n \leq z_n < \alpha + \varepsilon,$$

which is to say, that $|y_n - \alpha < \varepsilon$.

Consequently, $y_n \to \alpha$. ■

Theorem 17. *Let* $x_n \to x$. *If* $x_n \geq 0\ \forall n \in \mathbb{N}$, *then* $\sqrt{x_n} \to \sqrt{x}$.

Proof. Since $y_n \geq 0$ for all $n \in \mathbb{N}$, Theorem 15 implies that $y \geq 0$. There are 2 cases: $y = 0$ or $y > 0$.

(a) If $y = 0$, let $\varepsilon > 0$. Then $\varepsilon^2 > 0$. Since $y_n \to 0$, $\exists M_{\varepsilon^2} \in \mathbb{N}$ s.t. whenever $n > M_{\varepsilon^2}$, we must have $|y_n - 0| = y_n < \varepsilon^2$. Now, set $N_\varepsilon = M_{\varepsilon^2}$.

Then whenever $n > N_\varepsilon$, $|\sqrt{y_n} - 0| = \sqrt{y_n} < \sqrt{\varepsilon^2} = \varepsilon$.

(b) If $y > 0$, let $\varepsilon > 0$. Then $\varepsilon\sqrt{y} > 0$. Since $y_n \to y$, $\exists M_{\varepsilon\sqrt{y}} \in \mathbb{N}$ s.t. whenever $n > M_{\varepsilon\sqrt{y}}$, $|y_n - y| < \varepsilon\sqrt{y}$. Now, set $N_\varepsilon = M_{\varepsilon\sqrt{y}}$.

Then whenever $n > N_\varepsilon$, $|\sqrt{y_n} - \sqrt{y}| = \frac{|y_n - y|}{\sqrt{y_n} + \sqrt{y}} \leq \frac{|y_n - y|}{\sqrt{y}} < \frac{\varepsilon\sqrt{y}}{\sqrt{y}} = \varepsilon$.

In both cases, this yields $\sqrt{y_n} \to \sqrt{y}$. ■

Theorem 18. (Bounded Monotone Convergence)
Let (x_n) be an increasing sequence, bounded above. Then (x_n) converges to $\sup\{x_n \mid n \in \mathbb{N}\}$.

Proof. https://youtu.be/ZMCp9GzDmD8

Theorem 32. *If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$, then f is bounded on $[a, b]$.*

Proof. Suppose f is not bounded on $[a,b]$. Hence, for all $n \in \mathbb{N}$, $\exists x_n \in [a,b]$ such that $|f(x_n)| > n$. However, $(x_n) \subseteq [a,b]$ so that (x_n) is bounded.

By the BW Theorem, $\exists (x_{n_k}) \subseteq (x_n)$ such that $x_{n_k} \to \hat{x} \in [a,b]$, since

$$a \le x_{n_k} \le b \quad \text{for all } k.$$

Since f is continuous, we have

$$f(\hat{x}) = \lim_{x \to \hat{x}} f(x) = \lim_{k \to \infty} f(x_{n_k}),$$

so $(f(x_{n_k}))$ is bounded, being a convergent sequence. But this contradicts the assumption that $|f(x_{n_k})| > n_k \ge k$ for all k.

Hence f is bounded on $[a,b]$. ■

Theorem 33. (MAX/MIN THEOREM)
If $f : [a, b] \to \mathbb{R}$ is continuous, then f reaches a global maximum and a global minimum of $[a, b]$.

Proof. Let $f([a,b]) = \{f(x) \mid x \in [a,b]\}$. According to Theorem 32, $f([a,b])$ is bounded as f is continous, and so, by completeness of $\mathbb{R}$,

$$s^* = \sup\{f(x) \mid x \in [a,b]\} \quad \text{and} \quad s_* = \inf\{f(x) \mid x \in [a,b]\}$$

both exist.

We need only show $\exists x^*, x_* \in [a,b]$ such that $f(x^*) = s^*$ and $f(x_*) = s_*$.

Since $s^* - \frac{1}{n}$ is not an upper bound of $f([a,b])$ for every $n \in \mathbb{N}$, $\exists x_n \in [a,b]$ with

$$s^* - \frac{1}{n} < f(x_n) \leq s^*, \quad \text{for all } n \in \mathbb{N}.$$

According to the Squeeze Theorem, we must have $f(x_n) \to s^*$ (this says nothing about whether x_n converges or not, however).

But $(x_n) \subseteq [a, b]$ is bounded, so applying the BW Theorem, we find that $\exists (x_{n_k}) \subseteq (x_n)$ such that $x_{n_k} \to x^* \in [a, b]$.

As f is continuous,

$$s^* = \lim_{k \to \infty} f(x_{n_k}) = f\left(\lim_{k \to \infty} x_{n_k} \right) = f(x^*).$$

The existence of $x_* \in [a, b]$ such that $f(x_*) = s_*$ is shown similarly. ■

Theorem 34. *Let $f : [a, b] \to \mathbb{R}$ be continuous. If $\exists \alpha, \beta \in [a, b]$ such that $f(\alpha)f(\beta) < 0$, then $\exists \gamma \in (a, b)$ such that $f(\gamma) = 0$.*

Proof. We prove that the results holds for $f(\alpha)<0<f(\beta)$; the other case having a similar proof.

Write $\alpha_1=\alpha$, $\beta_1=\beta$, $I_1=[\alpha_1,\beta_1]$, and $\gamma_1=\frac{\alpha_1+\beta_1}{2}$. There are 3 possibilities:

i. if $f(\gamma_1)=0$, set $\gamma=\gamma_1$; then $\gamma\in(\alpha_1,\beta_1)$ and the theorem is proven;

ii. if $f(\gamma_1)>0$, set $\alpha_2=\alpha_1$, $\beta_2=\gamma_1$;

iii. if $f(\gamma_1)<0$, set $\alpha_2=\gamma_1$, $\beta_2=\beta_1$.

In the last two cases, set $I_2=[\alpha_2,\beta_2]$. Then $I_1\supseteq I_2$, length$(I_1)=\frac{\beta_1-\alpha_1}{2^0}$ and

$$f(\alpha_2)<0<f(\beta_2).$$

This is the base case $n = 1$ of an induction process, which can be extended for all $n \in \mathbb{N}$. One of two things can occur: either

1. $\exists n \in \mathbb{N}$ such that $f(\gamma_n) = 0$, with $\gamma_n \in (\alpha_n, \beta_n) \subseteq (\alpha, \beta)$, in which case the theorem is proven, or

2. there is a chain of nested intervals

$$I_1 \supseteq I_2 \supseteq \cdots I_k \supseteq I_{k+1} \supseteq \cdots$$

where $I_n = [\alpha_n, \beta_n]$, length$(I_n) = \frac{\beta_n - \alpha_n}{2^{n-1}}$, $f(\alpha_n) < 0 < f(\beta_n)\ \forall n \in \mathbb{N}$.

According to the Nested Intervals Theorem, since

$$\inf_{n \in \mathbb{N}} \{\text{length}(I_n)\} = \lim_{n \to \infty} \frac{\beta_n - \alpha_n}{2^{n-1}} = 0,$$

$\exists c \in [\alpha, \beta] \subseteq [a, b]$ such that $\bigcap_{n\in\mathbb{N}} I_n = \{c\}$.

It remains to show that $f(c) = 0$.

Note that the sequences $(\alpha_n), (\beta_n)$ both converge to c. Indeed, let $\varepsilon > 0$. By the Archimedean Property, $\exists N_\varepsilon \in \mathbb{N}$ such that $N_\varepsilon > \log_2(\frac{\beta-\alpha}{\varepsilon}) + 1$.

Since $c \in I_n$ for all $n \in \mathbb{N}$, then

$$|\alpha_n - c| < \text{length}(I_n) = \frac{\beta - \alpha}{2^{n-1}} < \varepsilon$$

whenever $n > N_\varepsilon$. The proof that $\beta_n \to c$ is identical.

Since f is continuous on $[a,b]$, it is also continuous at c. Thus,

$$\lim_{n\to\infty} f(\alpha_n) = \lim_{n\to\infty} f(\beta_n) = f(c).$$

But $f(\alpha_n) < 0$ for all n, so

$$f(c) = \lim_{n\to\infty} f(\alpha_n) \leq 0,$$

by Theorem 15. Using the same Theorem, we have $f(c) \geq 0$. Then $f(c) = 0$.

Lastly, note that $c \neq \alpha, \beta$; otherwise, $f(\alpha)f(\beta) = 0$.

This conclude the proof, with $\gamma = c$. ■

Theorem 35. (Intermediate Value Theorem)
Let $f : [a,b] \to \mathbb{R}$ *be continuous. If* $\exists \alpha < \beta \in [a,b]$ *s.t.* $f(\alpha) < k < f(\beta)$ *or* $f(\alpha) > k > f(\beta)$*, then* $\exists \gamma \in (a,b)$ *such that* $f(\gamma) = k$.

Proof. Assume that $f(\alpha) < k < f(\beta)$; the proof for the other case is similar.

Consider the function $g : [a, b] \to \mathbb{R}$ defined by $g(x) = f(x) - k$. Theorem 30 shows that g is continuous on $[a, b]$.

Furthermore,

$$g(\alpha) = f(\alpha) - k < k - k = 0 < f(\beta) - k = g(\beta).$$

According to Theorem 34, $\exists \gamma \in (\alpha, \beta)$ such that $g(\gamma) = f(\gamma) - k = 0$. Thus $f(\gamma) = k$. $\blacksquare$

Theorem 36. *If* $f : [a, b] \to \mathbb{R}$ *is continuous, then* $f([a, b])$ *is a closed and bounded interval.*

Proof. Let $m = \inf\{f[a,b]\}$ and $M = \sup\{f[a,b]\}$.

According to the Max/Min Theorem, $\exists \alpha, \beta \in [a,b]$ such that $f(\alpha) = m$ and $f(\beta) = M$.

If $m = M$, then f is constant and $f([a,b]) = [m,m] = [M,M]$.

If $m < M$, then $\alpha \neq \beta$. Furthermore, $m \leq f(x) \leq M$ for all $x \in [a,b]$, so that $f([a,b]) \subseteq [m,M]$.

Now, let $k \in [m,M]$. According to the Intermediate Value Theorem, $\exists \gamma$ between α and β such that $f(\gamma) = k$. Hence $k \in f([a,b])$ and so $[m,M] \subseteq f([a,b])$.

Consequently, $f([a,b]) = [m,M]$. ■

Theorem 37. *If $f : A \to \mathbb{R}$ is uniformly continuous on A, then f is continuous on A.*

Proof. Let $c \in A$ and $\varepsilon > 0$. As f is uniformly continuous on A, $\exists \delta_\varepsilon > 0$ such that

$$|f(x) - f(y)| < \varepsilon \quad \text{whenever } |x - y| < \delta_\varepsilon \text{ and } x, y \in A.$$

In particular, if $y = c$ then

$$|f(x) - f(c)| < \varepsilon \quad \text{whenever } |x - c| < \delta_\varepsilon \text{ and } x \in A.$$

As c is arbitrary, f is continuous on A. ■

Theorem 38. *Let $f : [a, b] \to \mathbb{R}$. Then f is uniformly continuous on $[a, b]$ if and only if f is continuous on $[a, b]$.*

Proof. Theorem 38 shows that if f is uniformly continuous on $[a, b]$, it is continuous on $[a, b]$.

Now, assume f is continuous on $[a, b]$. If f is not uniformly continuous, then $\exists \varepsilon_0 > 0$ such that $\forall \delta > 0$, $\exists x_\delta, y_\delta \in [a, b]$ with

$$|f(x_\delta) - f(y_\delta)| \geq \varepsilon_0 \text{ and } |x_\delta - y_\delta| < \delta.$$

For $n \in \mathbb{N}$, let $\delta_n = \frac{1}{n}$. The corresponding sequences $(x_{\delta_n}), (y_{\delta_n})$ lie in $[a, b]$, with

$$|x_{\delta_n} - y_{\delta_n}| < \delta_n = \frac{1}{n} \quad \text{and} \quad |f(x_{\delta_n}) - f(y_{\delta_n})| \geq \varepsilon_0, \quad \forall n \in \mathbb{N}.$$

As (x_{δ_n}) is bounded, $\exists (x_{\delta_{n_k}}) \subseteq (x_{\delta_n})$ such that $x_{\delta_{n_k}} \to z$ with $k \to \infty$, according to the Bolazano-Weierstrass Theorem.

Furthermore, $z \in [a,b]$ according to Theorem 15.

The corresponding sequence $(y_{\delta_{n_k}})$ also converges to z subce

$$0 \leq |y_{\delta_{n_k}} - z| \leq |y_{\delta_{n_k}} - x_{\delta_{n_k}}| + |x_{\delta_{n_k}} - z| < \frac{1}{n_k} + |x_{\delta_{n_k}} - z|$$

according to the Squeeze Theorem, as both $\frac{1}{n_k}, |x_{\delta_{n_k}} - z| \to 0$ with $k \to \infty$.

But f is continuous, both $(f(x_{\delta_{n_k}})), (f(y_{\delta_{n_k}})) \to f(z)$. But that is impossible as $|f(x_{\delta_n}) - f(y_{\delta_n})| \geq \varepsilon_0, \quad \forall n \in \mathbb{N}$.

Thus f must be uniformly continuous. ∎

Theorem 51. *Let* $I = [a, b]$ *and* f *be bounded on* I*. Then the lower integral and upper integral of* f *on* I *satisfy* $L(f) \leq U(f)$*.*

Proof. Let P_1, P_2 be partitions of I. Then $L(P_1; f) \leq U(P_2; f)$.

If we fix P_2, $U(P_2; f)$ is an upper bound for $\{L(P; f) \mid P \text{ a partition of } I\}$. As this set is bounded, its supremum $L(f)$ exists.

But P_2 was chosen arbitrarily, so $L(f)$ is a lower bound of

$$\{U(P; f) \mid P \text{ a partition of } I\}.$$

Consequently,

$$L(f) \leq \inf\{U(P; f) \mid P \text{ a partition of } I\} = U(f).$$

This completes the proof. ■

Theorem 52. (Riemann's Criterion)
Let $I = [a, b]$ and $f : I \to \mathbb{R}$ be a bounded function. Then f is Riemann-integrable if and only if $\forall \varepsilon > 0$, $\exists P_\varepsilon$ a partition of I such that the lower sum and the upper sum of f corresponding to P_ε satisfy $U(P_\varepsilon; f) - L(P_\varepsilon; f) < \varepsilon$.

Proof. If f is Riemann-integrable, then $L(f) = U(f) = \int_a^b f$.

Let $\varepsilon > 0$. Since $\int_a^b f - \frac{\varepsilon}{2}$ is not an upper bound of $\{L(P; f) \mid P \text{ a partition of } [a, b]\}$, there exists a partition P_1 such that

$$\int_a^b f - \frac{\varepsilon}{2} < L(P_1; f) \leq \int_a^b f.$$

Using a similar argument, there exists a partition P_2 such that

$$\int_a^b f + \frac{\varepsilon}{2} \geq U(P_2; f) > \int_a^b f.$$

Set $P_\varepsilon = P_1 \cup P_2$. Then P_ε is a refinement of P_1 and P_2.

Consequently,

$$\int_a^b f - \frac{\varepsilon}{2} < L(P_1; f) \leq L(P_\varepsilon; f) \leq U(P_\varepsilon; f) \leq U(P_2; f) < \int_a^b f + \frac{\varepsilon}{2}$$

which implies that

$$U(P_\varepsilon; f) - L(P_\varepsilon; f) < \varepsilon.$$

Conversely, let $\varepsilon > 0$ and P_ε be such that $U(P_\varepsilon; f) - L(P_\varepsilon; f) < \varepsilon$. Since $U(f) \leq U(P_\varepsilon; f)$ and $L(f) \geq L(P_\varepsilon; f)$, then

$$0 \leq U(f) - L(f) \leq U(P_\varepsilon; f) - L(P_\varepsilon; f) < \varepsilon.$$

But $\varepsilon > 0$ was arbitrary, so $U(f) - L(f) = 0$, which in turns implies that $U(f) = L(f)$ and that f is Riemann-integrable on $[a, b]$. ■

Theorem 53. *Let $I = [a, b]$ and $f : I \to \mathbb{R}$ be a monotone function on I. Then f is Riemann-integrable on I.*

Proof. We show that the result holds for increasing functions. A similar proof holds for decreasing functions.

Assume f is increasing. Let

$$P_n = \{x_i = a + i\left(\tfrac{b-a}{n}\right) : i = 0, \ldots, n\}$$

be the partition of $[a, b]$ into n equal sub-intervals. Since f is increasing on $[a, b]$, we have

$$\begin{aligned} m_i &= \inf\{f(x) \mid x \in [x_{i-1}, x_i]\} = f(x_{i-1}), \\ M_i &= \sup\{f(x) \mid x \in [x_{i-1}, x_i]\} = f(x_i), \end{aligned}$$

for $1 \leq i \leq n$.

Hence,

$$
\begin{aligned}
U(P_n;f)-L(P_n;f) &= \sum_{i=1}^{n} M_i(x_i-x_{i-1}) - \sum_{i=1}^{n} m_i(x_i-x_{i-1}) \\
&= \sum_{i=1}^{n}(M_i-m_i)(x_i-x_{i-1}) \\
&= \frac{b-a}{n}\sum_{i=1}^{n}(f(x_i)-f(x_{i-1})) \\
&= \frac{b-a}{n}\Big[f(x_1)-f(x_0)+\cdots+f(x_n)-f(x_{n-1})\Big] \\
&= \frac{b-a}{n}(f(b)-f(a)) \geq 0.
\end{aligned}
$$

Let $\varepsilon > 0$. By the Archimedean Property, $\exists N_\varepsilon \in \mathbb{N}$ such that

$$\frac{(b-a)(f(b)-f(a))}{\varepsilon} < n.$$

Set $P_\varepsilon = P_n$. Then

$$U(P_\varepsilon; f) - L(P_\varepsilon; f) < \frac{b-a}{N_\varepsilon}(f(b)-f(a)) < \varepsilon,$$

and f is Riemann-integrable on $[a,b]$ according to Riemann's Criterion. ■

Theorem 54. *Let* $f : [a, b] \to \mathbb{R}$ *be continuous, with* $a < b$. *Then* f *is Riemann-integrable on* $[a, b]$.

Proof. Let $\varepsilon > 0$.

According to Theorem 38, f is uniformly continuous on $[a, b]$. Hence $\exists \delta_\varepsilon > 0$ s.t. $|f(x) - f(y)| < \frac{\varepsilon}{b-a}$ whenever $|x - y| < \delta_\varepsilon$ and $x, y \in [a, b]$.

Pick $n \in \mathbb{N}$ such that $\frac{b-a}{n} < \delta_\varepsilon$ and let

$$P_\varepsilon = \{x_i = a + i\left(\tfrac{b-a}{n}\right) : i = 0, \ldots, n\}$$

be the partition of $[a, b]$ into n equal sub-intervals.

As f is continuous on $[x_{i-1}, x_i]$, $\exists u_i, v_i \in [x_{i-1}, x_i]$ such that

$$m_i = \inf\{f(x) \mid x \in [x_{i-1}, x_i]\} = f(u_i),$$
$$M_i = \sup\{f(x) \mid x \in [x_{i-1}, x_i]\} = f(v_i),$$

for all $1 \leq i \leq n$, according to the Max/Min Theorem. (Note that $|u_i - v_i| \leq \frac{b-a}{n} < \delta_\varepsilon$ for all i.)

Hence,

$$U(P_\varepsilon; f) - L(P_\varepsilon; f) = \sum_{i=1}^{n}(M_i - m_i)(x_i - x_{i-1}) = \frac{b-a}{n}\sum_{i=1}^{n}(f(v_i) - f(u_i))$$
$$< \frac{b-a}{n}\sum_{i=1}^{n}\frac{\varepsilon}{b-a} = \varepsilon,$$

by uniform continuity of f.

According to Riemann's Criterion, f is thus Riemann-integrable. ■

Theorem 55. (Properties of the Riemann Integral)
Let $I=[a,b]$ and $f,g:I\to\mathbb{R}$ be Riemann-integrable on I. Then

(a) $f+g$ is Riemann-integrable on I, with $\int_a^b(f+g)=\int_a^b f+\int_a^b g$;

(b) if $k\in\mathbb{R}$, $k\cdot f$ is Riemann-integrable on I, with $\int_a^b k\cdot f=k\int_a^b f$;

(c) if $f(x)\leq g(x)\ \forall x\in I$, then $\int_a^b f\leq\int_a^b g$, and

(d) if $|f(x)|\leq K\ \forall x\in I$, then $\left|\int_a^b f\right|\leq K(b-a)$.

Proof. We use a variety of pre-existing results.

(a) Let $\varepsilon > 0$. Since f, g are Riemann-integrable, $\exists P_1, P_2$ partitions of I such that $U(P_1; f) - L(P_1; f) < \frac{\varepsilon}{2}$ and $U(P_2; g) - L(P_2; g) < \frac{\varepsilon}{2}$.

Set $P = P_1 \cup P_2$. Then P is a refinement of P_1 and P_2, and

$$\begin{aligned} U(P; f+g) &\leq U(P; f) + U(P; g) \\ &< L(P; f) + L(P; g) + \varepsilon \leq L(P; f+g) + \varepsilon, \end{aligned} \tag{5}$$

since, over non-empty subsets of I, we have

$$\begin{aligned} \inf\{f(x) + g(x)\} &\geq \inf\{f(x)\} + \inf\{g(x)\} \\ \sup\{f(x) + g(x)\} &\leq \sup\{f(x)\} + \sup\{g(x)\}. \end{aligned}$$

Hence $f + g$ is Riemann-integrable according to Riemann's Criterion.

Furthermore, we see from (5) that

$$\int_a^b (f+g) \leq U(P;f+g) < L(P;f)+L(P;g)+\varepsilon \leq \int_a^b f + \int_a^b g + \varepsilon$$

and

$$\int_a^b f + \int_a^b g \leq U(P;f)+U(P;g) < L(P;f+g)+\varepsilon \leq \int_a^b (f+g)+\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, $\int_a^b f + \int_a^b g \leq \int_a^b (f+g) \leq \int_a^b f + \int_a^b g$, from which we conclude that $\int_a^b (f+g) = \int_a^b f + \int_a^b g$.

(b) The proof for $k=0$ is trivial. We show that the result holds for $k<0$ (the proof for $k>0$ is similar).

Let $P = \{x_0, \ldots, x_n\}$ be a partition of I. Since $k < 0$, we have $\inf\{kf(x)\} = k\sup\{f(x)\}$ over non-empty subsets of I, and so we have $L(P; kf) = kU(P; f)$. In particular,

$$\begin{aligned} L(kf) &= \sup\{L(P; kf) \mid P \text{ a partition of } I\} \\ &= \sup\{kU(P; f) \mid P \text{ a partition of } I\} \\ &= k\inf\{U(P; f) \mid P \text{ a partition of } I\} = kU(f) \end{aligned}$$

Similarly, $U(P; kf) = kL(P; f)$ and $U(fk) = kL(f)$, so

$$L(fk) = \underbrace{kU(f) = kL(f)}_{\text{since } f \text{ is R-int.}} = U(kf).$$

Thus kf is Riemann-integrable on I and $\int_a^b kf = L(k) = kU(f) = \int_a^b f$.

(c) We start by showing that if $h : I \to \mathbb{R}$ is integrable on I and $h(x) \geq 0$ for all $x \in I$, then $\int_a^b h(x) \geq 0$.

Let $P_0 = \{a, b\} = \{x_0, x_1\}$ and $m_1 = \inf\{h(x) \mid x \in [a, b]\} \geq 0$. Then,

$$0 \leq m_1(b - a) = L(P_0; h) \leq L(P; h)$$

for any partition P of I, as $P \supseteq P_0$. But h is Riemann-integrable by assumption, thus

$$\int_a^b h = \sup\{L(P; h) \mid P \text{ a partition of } I\} \geq L(P_0; h) \geq 0.$$

Then, set $h = g - f$. By hypothesis, $h(x) = g(x) - f(x) \geq 0$. Then

$$\int_a^b h = \int_a^b (g - f) = \int_a^b g - \int_a^b f \geq 0,$$

which implies that $\int_a^b g \geq \int_a^b f$.

(d) Let $P_0 = \{a, b\} = \{x_0, x_1\}$. As always, set $m_1 = \inf\{f(x) \mid x \in [a, b]\}$, and $M_1 = \sup\{f(x) \mid x \in [a, b]\}$. Then for any parition P of I, we have

$$\begin{aligned} m_1(b - a) &= L(P_0; f) \leq L(P; f) \leq L(f) = \int_a^b f \\ &= U(f) \leq U(P; f) \leq U(P_0; f) = M_1(b - a). \end{aligned}$$

In particular,

$$m_1(b-a) \leq \int_a^b f \leq M_1(b-a).$$

Now, if $|f(x)| \leq K$ for all $x \in I$, then $-K \leq m_1$ and $M_1 \leq K$ so that

$$-K(b-a) \leq m_1(b-a) \leq \int_a^b f \leq M_1(b-a) \leq K(b-a),$$

so that $|\int_a^b f| \leq K(b-a)$. ■

Theorem 56. (ADDITIVITY THE RIEMANN INTEGRAL)
Let $I = [a,b]$, $c \in (a,b)$, and $f : I \to \mathbb{R}$ be bounded on I. Then f is Riemann-integrable on I if and only if it is Riemann-integrable on $I_1 = [a,c]$ and on $I_2 = [c,b]$. When that is the case, $\int_a^b f = \int_a^c f + \int_c^b f$.

Proof. We start by assuming that f is Riemann-integrable on I.

Let $\varepsilon > 0$. According to the Riemann Criterion, $\exists P_\varepsilon$ a partition of I such that $U(P_\varepsilon; f) - L(P_\varepsilon; f) < \varepsilon$. Now, set $P = P_\varepsilon \cup \{c\}$. Then P is a refinement of P_ε so that

$$U(P; f) - L(P; f) \leq U(P_\varepsilon; f) - L(P_\varepsilon; f) < \varepsilon.$$

Set $P_1 = P \cap I_1$ and $P_2 = P \cap I_2$. Then P_i is a partition of I_i, and

$$\begin{aligned}\varepsilon &> U(P; f) - L(P; f) \geq U(P_1; f) + U(P_2; f) - L(P_1; f) - L(P_2; f) \\ &= \big[U(P_1; f) - L(P_1; f)\big] + \big[U(P_2; f) - L(P_2; f)\big]\end{aligned}$$

Consequently, $U(P_i; f) - L(P_i; f) < \varepsilon$ for $i = 1, 2$ and f is Riemann-integrable on I_1 and I_2, according to the Riemann Criterion.

Now assume that f is Riemann-integrable on I_1 and I_2.

Let $\varepsilon > 0$. According to the Riemann Criterion, for $i = 1, 2$, $\exists P_i$ a partition of I_i such that

$$U(P_i; f) + L(P_i; f) < \frac{\varepsilon}{2}.$$

Set $P = P_1 \cup P_2$. Then P is a partition of I. Furthermore,

$$\begin{aligned} U(P; f) - L(P; f) &= U(P_1; f) + U(P_2; f) - L(P_1; f) - L(P_2; f) \\ &= U(P_1; f) - L(P_1; f) + U(P_2; f) - L(P_2; f) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \end{aligned}$$

thus f is Riemann-integrable on I according the Riemann Criterion.

Finally, let's assume that f is Riemann-integrable on I (and so on I_1, I_2), or vice-versa.

Let P_1, P_2 be partitions of I_1, I_2, respectively, such that

$$U(P_i; f) - L(P_i; f) < \frac{\varepsilon}{2}, \quad i = 1, 2.$$

Set $P = P_1 \cup P_2$. Then P is a partition of I and

$$\begin{aligned} \int_a^b f &\leq U(P; f) = U(P_1; f) + U(P_2; f) \\ &< L(P_1; f) + L(P_2; f) + \varepsilon = \int_a^c f + \int_c^b f + \varepsilon. \end{aligned}$$

Similarly,

$$\begin{aligned}\int_a^b f &\geq L(P;f) = L(P_1;f) + L(P_2;f) \\ &> U(P_1;f) + U(P_2;f) - \varepsilon \geq \int_a^c f + \int_c^b f - \varepsilon\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, $\int_a^b f = \int_a^c f + \int_c^b f$. $\blacksquare$

Theorem 57. (Composition Theorem for Integrals)
Let $I = [a,b]$ and $J = [\alpha,\beta]$, $f : I \to \mathbb{R}$ Riemann-integrable on I, $\varphi : J \to \mathbb{R}$ continuous on J and $f(I) \subseteq J$. Then $\varphi \circ f : I \to \mathbb{R}$ is Riemann-integrable on I.

Proof. Let $\varepsilon > 0$, $K = \sup\{|\varphi(x)| \mid x \in J\}$ (guaranteed to exist by the Max/Min theorem) and $\varepsilon' = \frac{\varepsilon}{b-a+2K}$.

Since φ is uniformly continuous on J (being continuous on a closed, bounded interval), $\exists \delta_\varepsilon > 0$ s.t.

$$|x-y| < \delta_\varepsilon,\ x, y, \in J \implies |\varphi(x) - \varphi(y)| < \varepsilon'.$$

Without loss of generality, pick $\delta_\varepsilon < \varepsilon'$.

Since f is Riemann-integrable on I, $\exists P = \{x_0, \ldots, x_n\}$ a partition of $I = [a, b]$ s.t.

$$U(P; f) - L(P; f) < \delta_\varepsilon^2$$

(according to Riemann's criterion).

We show that $U(P; \varphi \circ f) - L(P; \varphi \circ f) < \varepsilon$, and so that $\varphi \circ f$ is Riemann-integrable according to Riemann's criterion.

Over $[x_{i-1}, x_i]$ for $i = 1, \ldots, n$, set

$$m_i = \inf\{f(x)\},\ M_i = \sup\{f(x)\},\ \tilde{m}_i = \inf\{\varphi(f(x))\},\ \tilde{M}_i = \sup\{\varphi(f(x))\}.$$

With those, set $A = \{i \mid M_i - m_i < \delta_\varepsilon\}$, $B = \{i \mid M_i - m_i \geq \delta_\varepsilon\}$.

- If $i \in A$, then

$$x, y \in [x_{i-1}, x_i] \implies |f(x) - f(y)| \leq M_i - m_i < \delta_\varepsilon,$$

so $|\varphi(f(x)) - \varphi(f(y)| < \varepsilon' \; \forall x, y \in [x_{i-1}, x_i]$. In particular, $\tilde{M}_i - \tilde{m}_i \leq \varepsilon'$.

- If $i \in B$, then

$$x, y \in [x_{i-1}, x_i] \implies |\varphi(f(x)) - \varphi(f(y))| \leq |\varphi(f(x))| + |\varphi(f(y))| \leq 2K.$$

In particular, $\tilde{M}_i - \tilde{m}_i \leq 2K$, since $-K \leq \tilde{m}_i \leq \varphi(z) \leq \tilde{M}_i \leq K$ for all $z \in [x_{i-1}, x_i]$.

Then

$$\begin{aligned}
U(P;\varphi\circ f)-L(P;\varphi\circ f) &= \sum_{i=1}^{n}(\tilde{M}_i-\tilde{m}_i)(x_i-x_{i-1}) \\
&= \sum_{i\in A}(\tilde{M}_i-\tilde{m}_i)(x_i-x_{i-1})+\sum_{i\in B}(\tilde{M}_i-\tilde{m}_i)(x_i-x_{i-1}) \\
&\leq \varepsilon'\sum_{i\in A}(x_i-x_{i-1})+2K\sum_{i\in B}(x_i-x_{i-1}) \\
&\leq \varepsilon'(b-a)+2K\sum_{i\in B}\frac{(M_i-m_i)}{\delta_\varepsilon}(x_i-x_{i-1}) \\
& \varepsilon'(b-a)+\frac{2K}{\delta_\varepsilon}\sum_{i=1}^{n}(M_i-m_i)(x_i-x_{i-1}).
\end{aligned}$$

By earlier work in the proof, we have

$$\sum_{i=1}^{n}(M_i - m_i)(x_i - x_{i-1}) \leq U(P;f) - L(P;f) < \delta_\varepsilon^2,$$

so that

$$\begin{aligned} U(P;\varphi\circ f) - L(P;\varphi\circ f) &< \varepsilon'(b-a) + \frac{2K}{\delta_\varepsilon}\cdot\delta_\varepsilon^2 \\ &= \varepsilon'(b-a) + 2K\delta_\varepsilon < \varepsilon'(b-a) + 2K\varepsilon' \\ &= \varepsilon'(b-a+2K) = \varepsilon, \end{aligned}$$

which completes the proof. ∎

Theorem 58. *Let* $I = [a, b]$ *and* $f, g : I \to \mathbb{R}$ *be Riemann-integrable on* I*. Then* fg *and* $|f|$ *are Riemann-integrable on* I*, and* $\left|\int_a^b f\right| \leq \int_a^b |f|$*.*

Proof. The function defined by $\varphi(t)=t^2$ is continuous. By the Composition Theorem, $\varphi\circ(f+g)=(f+g)^2$ and $\varphi\circ(f-g)=(f-g)^2$ are both Riemann-integrable on I.

But the product fg can be re-written as

$$fg=\frac{1}{4}\big[(f+g)^2-(f-g)^2\big].$$

According to Theorem 55, fg is Riemann-integrable on I (note that there is no general form for $\int_a^b fg$).

Now, consider the function defined by $\varphi(t)=|t|$. It is continuous, so $\varphi\circ f=|f|$ is R-integrable on I according to the Composition Theorem.

Pick $c \in \{\pm 1\}$ such that $c\int_a^b f \geq 0$. Hence

$$\left|\int_a^b f\right| = c\int_a^b f = \int_a^b cf \leq \int_a^b |f|,$$

since $cf(x) \leq |f(x)|$ for all $x \in I$. ■