**MAT 2377 – Assignment 2 – Solutions**

*Total* = 100 marks

**Solution 1.** [not marked] Let $X$ be the number of light bulbs that fail during the 24-hour test.

**(a)** $X \sim \mathcal{B}(n,p)$, with $n = 10$ and $p = 0.01$.

**(b)** We are looking for $P(X=0) = \binom{10}{0}(0.01)^0(0.99)^{10} = (0.99)^{10} \approx 0.904$.

**(c)** We are looking for

$$\begin{aligned}
P(X \geq 3) &= 1 - P(X=0) - P(X=1) - P(X=2) \\
&= 1 - \binom{10}{0}(0.01)^0(0.99)^{10} - \binom{10}{1}(0.01)^1(0.99)^9 - \binom{10}{2}(0.01)^2(0.99)^8 \\
&= 1 - (0.99)^{10} - 10(0.01)(0.99)^9 - 45(0.01)^2(0.99)^8 \approx 0.0001.
\end{aligned}$$

**(d)** We are looking for

$$\begin{aligned}
P(2 \leq X \leq 4) &= P(X=2) + P(X=3) + P(X=4) \\
&= \binom{10}{2}(0.01)^2(0.99)^8 + \binom{10}{3}(0.01)^3(0.99)^7 + \binom{10}{4}(0.01)^4(0.99)^6 \approx 0.0043.
\end{aligned}$$

**Solution 2.** [30 marks] Let $T_1$ and $T_2$ be the number of samples required to get 1 and 2 positive algae bloom tests, respectively.

**(a)** [10 marks] $T_1 \sim \text{Geo}(0.03)$. We are looking for $P(T_1 = 7) = (1-0.03)^6(0.03) \approx 0.0250$.

**(b)** [10 marks] We have $T_2 \sim \text{NegBinom}(2, 0.03)$, with $\text{E}[T_2] = \frac{2}{0.03} \approx 66.7$ and $\text{SD}[T_2] = \sqrt{\frac{2(1-0.03)}{(0.03)^2}} \approx 46.4280$

**(c)** [10 marks] We are looking for $P(T_2 = 5) = \binom{5-1}{2-1}(0.03)^2(1-0.03)^3 \approx 0.0033$.

**Solution 3.** [40 marks] Let $N_{100}$ be the number of defects over a 100m stretch of tape. Since $\lambda = 2$ defects per 100m, $N_{100} \sim \text{Poisson}(2)$.

**(a)** [10 marks] We have $P(N_{100} = x) = \frac{2^x e^{-2}}{x!}$, $x = 0, 1, \ldots$.

**(b)** [10 marks] We have $\text{E}[N_{100}] = \lambda = 2$ and $\text{SD}[N_{100}] = \sqrt{\lambda} = \sqrt{2}$ defects per 100m.

**(c)** [10 marks] Let $N_{300}$ be the number of defects over a 300m stretch of tape. 300m is $3 \times 100$m, so $\text{E}[N_{300}] = 3\lambda = 3(2) = 6$ defects per 300m.

**(d)** [10 marks] The length of tape between 20 and 75 is 55m. The number of defects over that stretch of tape follows a Poisson distribution with parameter $2 \cdot 0.55 = 1.1$, i.e. $N_{55} \sim \text{Poisson}(1.1)$. Thus

$$P(N_{55} > 2) = 1 - P(N_{55} \leq 2) = 1 - \frac{(1.1)^0 e^{-1.1}}{0!} - \frac{(1.1)^1 e^{-1.1}}{1!} - \frac{(1.1)^2 e^{-1.1}}{2!} = 0.0996.$$

**Solution 4.** [30 marks] Let $X$ be the lifetime of a helicopter part. We have $3.2 = \mathrm{E}[X] = \frac{1}{\lambda}$, whence $\lambda = 1/3.2$. The probability distribution of this exponential random variable is

$$f_X(x) = \frac{1}{3.2}e^{-x/3.2}, \text{ when } x \geq 0 \text{ and } 0 \text{ otherwise;}$$

the cumulative distribution function is

$$F_X(x) = 1 - e^{-x/3.2} \text{ when } x \geq 0 \text{ and } 0 \text{ otherwise.}$$

**(a)** [10 marks] We are looking for

$$P(X > 4.4) = 1 - P(X < 4.4) = 1 - F_X(4.4) = 1 - (1 - e^{-4.4/3.2}) \approx 0.2528.$$

**(b)** [10 marks] 3 months is 0.25 year, and 9 months is 0.75 year, so we are looking for

$$\begin{aligned} P(0.25 < X < 0.75) &= F_X(0.75) - F_X(0.25) \\ &= (1 - e^{-0.75/3.2}) - (1 - e^{-0.25/3.2}) \approx 0.1338 \end{aligned}$$

**(c)** [10 marks] Since the exponential distribution is memory-less, we are looking for

$$P(X \geq 5|X > 3) = P(X > 5 - 3) = P(X > 2) = e^{-2/3.2} \approx 0.5353.$$

**Solution 5.** [not marked] Let $W_1$ the time between two successive cars passing the location, with the number of cars passing the location each minute being a Poisson process with parameter $\lambda = 5$.

**(a)** Then $W_1 \sim \mathrm{Exp}(5)$, and $\mathrm{E}[W_1] = \frac{1}{5}$ and $\mathrm{Var}[W_1] = \frac{1}{5^2} = \frac{1}{25}$.

**(b)** We are looking for $P(W_1 > 1) = 1 - P(W_1 < 1) = 1 - F_{W_1}(1) = e^{-5} = 0.0067$.

**(c)** Let $W_2$ be the wait time for 2 cars to pass by the location. Then $W_2 \sim \Gamma(5, 2)$ and

$$P(W_2 > 1) = \int_1^\infty \frac{w^{2-1}}{(2-1)!}5^2 e^{-5w}\,dw = \int_1^\infty 25we^{-5w} \approx 0.0404.$$

**Solution 6.** [not marked] By assumptions, $X \sim N(10, 0.02^2)$.

**(a)** We are looking for

$$P(X \leq 9.97) = P\left(\frac{X - 10}{0.02} \leq \frac{9.97 - 10}{0.02}\right) = \Phi\left(\frac{9.97 - 10}{0.02}\right) = \Phi(-1.5) \approx 0.0668,$$

so about 6.7%.

**(b)** We are looking for

$$\begin{aligned} P(X \geq 10.05) = 1 - P(X \geq 10.05) &= 1 - P\left(\frac{X - 10}{0.02} \leq \frac{10.05 - 10}{0.02}\right) \\ &= 1 - \Phi\left(\frac{10.05 - 10}{0.02}\right) = 1 - \Phi(2.5) \approx 1 - 0.9938 = 0.0062, \end{aligned}$$

so about 0.6%.

**(c)** We are looking for $P(X \geq 9.97 \text{ or } X \leq 10.03)$. But

$$\begin{aligned} P(9.97 \leq X \leq 10.03) &= P\left(\frac{9.97-10}{0.02} \leq \frac{X-10}{0.02} \leq \frac{10.03-10}{0.02}\right) \\ &= \Phi\left(\frac{10.03-10}{0.02}\right) - \Phi\left(\frac{9.97-10}{0.02}\right) = \Phi(1.5) - \Phi(-1.5) \approx 0.8664, \end{aligned}$$

so $P(X \geq 9.97 \text{ or } X \leq 10.03) = 1 - P(9.97 \leq X \leq 10.03) \approx 0.1336$ or about 13.3%.

**(d)** We are looking for a $c$ such that $1 - P(10 - c \leq X \leq 10 + c) = 0.05$; this is equivalent to

$$\begin{aligned} P(10-c \leq X \leq 10+c) = 0.95 &\iff P\left(\frac{10-c-10}{0.02} \leq \frac{X-10}{0.02} \leq \frac{10-c-10}{0.02}\right) = 0.95 \\ &\iff P\left(-50c \leq Z \leq 50c\right) = 0.95 \\ &\iff \Phi(50c) - \Phi(-50c) = 0.95 \\ &\iff 2\Phi(50c) - 1 = 0.95 \\ &\iff \Phi(50c) = 0.975 \\ &\iff 50c = 1.96 \iff c = 1.96/50 \approx 0.039. \end{aligned}$$

Thus $P(9.961 \leq X \leq 10.039) \approx 0.95$ and $P(X \leq 9.961 \text{ or } X \leq 10.039) \approx 0.05$.

**(e)** If $X \sim N(10.01, 0.02^2)$, then

$$\begin{aligned} P(9.961 \leq X \leq 10.039) &= P\left(\frac{9.961-10.01}{0.02} \leq \frac{X-10.01}{0.02} \leq \frac{10.039-10.01}{0.02}\right) \\ &= \Phi\left(\frac{10.039-10.01}{0.02}\right) - \Phi\left(\frac{9.961-10.01}{0.02}\right) \\ &= \Phi(1.45) - \Phi(-2.45) \approx 0.9265 - 0.0071 = 0.9194, \end{aligned}$$

or about 92%.