MAT 2125 – Homework 5 – Solutions

(due at midnight on April 14, in Brightspace)

1 Properties of the Riemann Integral

1. Let $f : [a,b] \to \mathbb{R}$ be continuous, $f \geq 0$ on $[a,b]$, and $\int_a^b f = 0$. Show that $f(x) = 0$ for all $x \in [a,b]$.

   Proof: We will show the contrapositive of the statement.

   Suppose that there exists $z \in [a,b]$ such that $f(z) > 0$. Since f is continuous, we may assume $z \in (a,b)$.[1] Then, taking $\varepsilon = f(z)/2$ in the definition of continuity, there exists a $\delta > 0$ such that

   $$|x - z| < \delta \implies |f(x) - f(z)| < f(z)/2 \implies f(x) > f(z)/2.$$

   Reducing δ if necessary, we may assume $\delta \leq \min\{z-a, b-a\}$. Therefore,

   $$[z - \delta/2, z + \delta/2] \subseteq (z - \delta, z + \delta) \subseteq [a,b].$$

   Thus

   $$\int_a^b f = \int_a^{z-\delta/2} f + \int_{z-\delta/2}^{z+\delta/2} f + \int_{z+\delta/2}^b f \geq 0 + \delta f(z)/2 + 0 > 0.$$

   This completes the proof. ■

2. Let $f : [a,b] \to \mathbb{R}$ be continuous and let $\int_a^b f = 0$. Show $\exists c \in [a,b]$ such that $f(c) = 0$.

   Proof: We will show the contrapositive of the statement.

   Suppose $f(c) \neq 0$ for all $c \in [a,b]$. Then, by the Intermediate Value Theorem, either $f(x) > 0$ for all $x \in [a,b]$ or $f(x) < 0$ for all $x \in [a,b]$.

   If $f(x) > 0$ for all $x \in [a,b]$, then $\int_a^b f > 0$ by the preceding question. Similarly, if $f(x) < 0$ for all $x \in [a,b]$, then $\int_a^b (-f) > 0$, which implies that $-\int_a^b f > 0$. In both cases, $\int_a^b f \neq 0$. ■

2 Fundamental Theorem of Calculus

1. Let $f : [0,3] \to \mathbb{R}$ be defined by

   $$f(x) = \begin{cases} x & x \in [0,1) \\ 1 & x \in [1,2) \\ x & x \in [2,3] \end{cases}.$$

   Find $F : [0,3] \to \mathbb{R}$, where

   $$F(x) = \int_0^x f.$$

   Where is F differentiable? What is F' there?

   Proof: The function f is increasing on $[0,3]$ so it is Riemann integrable there. The function F is given by

   $$F(x) = \begin{cases} \frac{x^2}{2}, & x \in [0,1) \\ x - \frac{1}{2}, & x \in [1,2) \\ \frac{x^2-1}{2}, & x \in [2,3] \end{cases}$$

   By the Fundamental Theorem of Calculus, F is differentiable wherever f is continuous, that is on $[0,2) \cup (2,3]$, and $F' = f$ there. ■

[1] If $f(z) = 0$ for all $z \in (a,b)$, then $f(a) = f(b) = 0$.

2. Compute $\dfrac{d}{dx}\displaystyle\int_{-x}^{x} e^{t^2}\,dt$.

   Proof: According to the additivity property of the Riemann integral and the Fundamental Theorem of Calculus, we have

$$\begin{aligned}\frac{d}{dx}\int_{-x}^{x} e^{t^2}\,dt &= \frac{d}{dx}\left(\int_{-x}^{0} e^{t^2}\,dt + \int_{0}^{x} e^{t^2}\,dt\right) = \frac{d}{dx}\left(-\int_{0}^{-x} e^{t^2}\,dt + \int_{0}^{x} e^{t^2}\,dt\right)\\ &= -\frac{d}{dx}\int_{0}^{-x} e^{t^2}\,dt + \frac{d}{dx}\int_{0}^{x} e^{t^2}\,dt = -e^{x^2}\cdot(-1) + e^{x^2} = 2e^{x^2},\end{aligned}$$

   where we used the chain rule in the second-to-last equality. ■

3 Improper Integrals

1. Let $f : [a, b] \to \mathbb{R}$ be Riemann-integrable on $[a + \delta, b]$ and unbounded in the interval $(a, a + \delta)$ for every $0 < \delta < b - a$. Define

$$\int_a^b f = \lim_{\delta\to 0^+} \int_{a+\delta}^{b} f,$$

   where $\delta \to 0^+$ means that $\delta \to 0$ and $\delta > 0$. A similar construction allows us to define

$$\int_a^b g = \lim_{\delta\to 0^+} \int_{a}^{b-\delta} g.$$

   Such integrals are said to be *improper*; when the limits exist, they are further said to be *convergent*.

   How can the expression

$$\int_0^1 \frac{1}{\sqrt{|x|}}\,dx$$

   be interpreted as an improper integral? Is it convergent? If so, what is its value?

   Proof: By definition,

$$\int_0^1 \frac{1}{\sqrt{|x|}}\,dx = \lim_{a\to 0^+}\int_a^1 \frac{1}{\sqrt{x}}\,dx = \lim_{a\to 0^+}\left(2\sqrt{1} - 2\sqrt{a}\right) = 2.$$

   Thus the improper integral converges to 2. ■

2. For which values of s does the integral $\int_0^1 x^s\,dx$ converge? You may use the antiderivatives rules of calculus.

   Proof. If $s \geq 0$, the integral is not improper. The integrand being continuous, the integral exists (i.e. converges) for those values of s. Now, consider $s < 0$.

   First assume $s \neq -1$. We have

$$\int_0^1 x^s\,dx = \lim_{a\to 0^+}\int_a^1 x^s\,dx = \lim_{a\to 0^+}\left(\frac{1^{s+1}}{s+1} - \frac{a^{s+1}}{s+1}\right).$$

   This limit exists if and only if $s > -1$, in which case it is equal to $\frac{1}{s+1}$. Now, if $s = -1$, then we have

$$\int_0^1 x^s\,dx = \lim_{a\to 0^+}\int_a^1 x^{-1}\,dx = \lim_{a\to 0^+}\left(\log 1 - \log a\right),$$

   which does not exist. Therefore, the given improper integral converges if and only if $s > -1$. ■

4 Sequences of Functions

1. Show that $\displaystyle\lim_{n\to\infty}\int_{\pi/2}^{\pi}\frac{\sin(nx)}{nx}\,dx = 0$. You may assume that sin is continuous and that $|\sin x| \le 1$, $\forall x \in \mathbb{R}$.

   Proof: For $n \in \mathbb{N}$, define $f_n : [\pi/2, \pi] \to \mathbb{R}$ by
   $$f_n(x) = \frac{\sin(nx)}{nx}.$$
   Then each f_n is continuous. For all $n \in \mathbb{N}$, we have
   $$\sup_{x\in[\pi/2,\pi]}\left\{\left|\frac{\sin(nx)}{nx}\right|\right\} \le \frac{2}{n\pi}.$$
   Since $2/n\pi \to 0$ as $n \to \infty$, we have $f_n \rightrightarrows 0$ (why?). Then the limit interchange theorem for integrals applies, and we have
   $$\lim_{n\to\infty}\int_{\pi/2}^{\pi}\frac{\sin(nx)}{nx}\,dx = \int_{\pi/2}^{\pi} 0\,dx = 0.$$
   This completes the proof. ■

2. Show that if $f_n \rightrightarrows f$ on $[a, b]$, and each f_n is continuous, then the sequence of functions
   $$F_n(x) = \int_a^x f_n(t)\,dt$$
   also converges uniformly on $[a, b]$.

   Proof: Define $F(x) = \int_a^x f(t)\,dt$. Let $\varepsilon > 0$. Since $f_n \rightrightarrows f$, $\exists N \in \mathbb{N}$ such that, for all $n \ge N$, we have
   $$|f_n(x) - f(x)| < \frac{\varepsilon}{b-a} \quad \forall\, x \in [a, b].$$
   Then, for all $n \ge N$ and $x \in [a, b]$, we have
   $$|F_n(x) - F(x)| = \left|\int_a^x f_n(t)\,dt - \int_a^x f(t)\,dt\right| \le \int_a^x |f_n(t) - f(t)|\,dt \le (x-a)\cdot\frac{\varepsilon}{b-a} \le \varepsilon.$$
   Thus $F_n \rightrightarrows F$ on $[a, b]$. ■

5 Series and Power Series

1. If the series $\displaystyle\sum_{k=1}^{\infty} a_k$ and $\displaystyle\sum_{k=1}^{\infty} b_k$ both converge, does $\displaystyle\sum_{k=1}^{\infty} a_k b_k$ converge?

   Proof: In general, we cannot conclude that $\displaystyle\sum_{k=1}^{\infty} a_k b_k$ converges. For instance, if
   $$a_k = b_k = \frac{(-1)^{k-1}}{\sqrt{k}},$$
   then $\displaystyle\sum_{k=1}^{\infty} a_k$ and $\displaystyle\sum_{k=1}^{\infty} b_k$ both converge by the Alternating Series Test. However, $\displaystyle\sum_{k=1}^{\infty} a_k b_k$ is the harmonic series, which diverges.

   Now assume that $a_k, b_k \ge 0$ for all $k \in \mathbb{N}$. We will show that $\displaystyle\sum_{k=1}^{\infty} a_k b_k$ converges. Since $\displaystyle\sum_{k=1}^{\infty} b_k$ converges, we have $b_k \to 0$. Therefore, there exists $N \in \mathbb{N}$ such that
   $$0 \le b_n < 1 \quad \forall\, n \ge N.$$

Thus, for all $n \geq N$, we have

$$0 \leq a_n b_n \leq a_n.$$

Then $0 \leq \sum_{k=1}^{\infty} a_k b_k \leq \sum_{k=1}^{\infty} a_k$, and as the latter converges, so does the former. ■

2. Find the radius of convergence for each of the following series.

   (a) $\sum_{k=0}^{\infty} (-1)^k x^{2k}$.

   (b) $\sum_{k=0}^{\infty} k x^k$.

   (c) $\sum_{k=0}^{\infty} k! x^k$.

Solution: It what follows, let a_k be the coefficient of x^k in the given power series.

(a) We have

$$\sqrt[k]{|a_k|} = \begin{cases} 1 & \text{if } k \text{ is even,} \\ 0 & \text{if } k \text{ is odd.} \end{cases}$$

Thus $\limsup_{k\to\infty} \sqrt[k]{|a_k|} = 1$, and so $R = 1$.

(b) Using one of the exercises done in class, we have

$$\sqrt[k]{|a_k|} = \sqrt[k]{k} \to 1 \quad \text{as} \quad k \to \infty.$$

Thus, $R = 1$.

(c) It can be shown that $n! > (n/2)^{n/2}$. Therefore,

$$\sqrt[n]{n!} > \left(\frac{n}{2}\right)^{1/2}.$$

Since $\sqrt{n/2} \to \infty$ as $n \to \infty$, we have $\sqrt[n]{n!} \to \infty$ as $n \to \infty$. Therefore, $R = 0$. ■