MAT 2125
Elementary Real Analysis

Exercises – Solutions – Q84-Q87

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84. Suppose $f : [0, \infty) \to \mathbb{R}$ is continuous and $f(x) \neq 0$ for all $x > 0$. If

$$(f(x))^2 = 2\int_0^x f \quad \text{for all } x > 0,$$

show that $f(x) = x$ for all $x \geq 0$.

Proof. As f is continuous, $F(x) = \int_0^x f$ is continuous and $F'(x) = f(x)$ for all $x \in [0, \infty)$ by the Fundamental Theorem of Calculus (2nd version),

Either $f(x) > 0$ for all $x > 0$ or $f(x) < 0$ for all $x > 0$ – otherwise f would admit a root $c > 0$ by the IVT, which contradicts $f(x) \neq 0 \, \forall x > 0$.

But

$$F(x) = \int_0^x f = \frac{(f(x))^2}{2} > 0 \quad \text{for all } x > 0,$$

so $\int_0^x f > 0$ for all $x > 0$, i.e. $f > 0$ for all $x > 0$ – otherwise, $\int_0^x f \leq \int_0^x 0 = 0$, which contradicts one of the above inequalities.

By construction,

$$\frac{(f(0))^2}{2} = F(0) = \int_0^0 f = 0,$$

that is $f(0)=0$. Now, let $c>0$. By hypothesis, $F'(c)=f(c)>0$. Furthermore, $F(c)=\frac{(f(c))^2}{2}$. As f is continuous at c,

$$\lim_{x\to c}\frac{1}{2}\left(f(x)+f(c)\right)=f(c).$$

Thus we have:

$$1=\frac{F'(c)}{f(c)}=\frac{\displaystyle\lim_{x\to c}\frac{F(x)-F(c)}{x-c}}{\displaystyle\lim_{x\to c}\frac{1}{2}\left(f(x)+f(c)\right)}=\lim_{x\to c}\frac{\left(f(x)\right)^2-\left(f(c)\right)^2}{(x-c)\left(f(x)+f(c)\right)}$$

$$=\lim_{x\to c}\frac{\left(f(x)-f(c)\right)\left(f(x)+f(c)\right)}{(x-c)\left(f(x)+f(c)\right)}=\lim_{x\to c}\frac{f(x)-f(c)}{x-c}=f'(c).$$

Hence f is differentiable and $f'(c) = 1$ for all $c > 0$. Then, by the Fundamental Theorem of Calculus (1st version),

$$\int_0^x f' = f(x) - f(0) = f(x) - 0 = f(x)$$

for all $x \geq 0$. As $\int_0^x f' = \int_0^x 1 = x - 0 = x$, this completes the proof, which, incidentally, is one of my favourite proof in the course. ■

85. Let $f, g : [a, b] \to \mathbb{R}$ be continuous and s.t.

$$\int_a^b f = \int_a^b g.$$

Show that there exists $c \in [a, b]$ s.t. $f(c) = g(c)$.

Proof. As f and g are continuous, the functions

$$F(x)=\int_a^x f \quad \text{and} \quad G(x)=\int_a^x g$$

are continuous and differentiable on $[a,b]$, with $F'(x)=f(x)$ and $G'(x)=g(x)$, according to the Fundamental Theorem of Calculus. Then $H(x)=F(x)-G(x)$ is continuous.

But by hypothesis,

$$H(a)=F(a)-G(a)=\int_a^a f-\int_a^a g=0-0=0$$

$$H(b)=F(b)-G(b)=\int_a^b f-\int_a^b g=0.$$

Since H is also differentiable, $\exists c \in (a, b)$ s.t. $H'(c) = 0$, by Rolle's Theorem. As

$$H'(c) = F'(c) - G'(c) = f(c) - g(c) = 0,$$

this completes the proof. ■

86. Let $f : [0, 3] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x & x \in [0, 1) \\ 1 & x \in [1, 2) \\ x & x \in [2, 3] \end{cases}.$$

Find $F : [0, 3] \to \mathbb{R}$, where

$$F(x) = \int_0^x f.$$

Where is F differentiable? What is F' there?

Proof. The function f is increasing on $[0,3]$ so it is Riemann integrable there. The function F is given by

$$F(x)=\begin{cases}\frac{x^2}{2}, & x\in[0,1)\\ x-\frac{1}{2}, & x\in[1,2)\\ \frac{x^2-1}{2}, & x\in[2,3]\end{cases}$$

By the Fundamental Theorem of Calculus, F is differentiable wherever f is continuous, that is on $[0,2)\cup(2,3]$, and $F'=f$ there. ■

87. If $f : [0, 1] \to \mathbb{R}$ is continuous and $\int_0^x f = \int_x^1 f$ for all $x \in [0, 1]$, show that $f \equiv 0$.

Proof. As f is continuous, then $F(x) = \int_0^x f$ is continuous and differentiable on $[0,1]$, with $F'(x) = f(x)$, by the Fundamental Theorem of Calculus.

By the Additivity Theorem,

$$\int_0^x f + \int_x^1 f = \int_0^1 f.$$

But $\int_0^x f = \int_x^1 f$ so $2\int_0^x f = \int_0^1 f$. In particular,

$$F(x) = \frac{1}{2}\int_0^1 f = \text{constant}.$$

Then $f(x) = F'(x) = 0$ for all $x \in [0,1]$. ■