MAT 2125
Elementary Real Analysis

Exercises – Solutions – Q88-Q92

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88. Show that $\lim_{n\to\infty} \dfrac{nx}{1+n^2x^2} = 0$ for all $x \in \mathbb{R}$.

Proof. If $x = 0$, then $\frac{nx}{1+n^2x^2} = 0 \to 0$.

If $x \neq 0$, let $\varepsilon > 0$. By the Archimedean property, $\exists N_\varepsilon > \frac{1}{\varepsilon|x|}$ (depending on x) s.t.

$$\left|\frac{nx}{1+n^2x^2} - 0\right| = \frac{n|x|}{1+n^2x^2} < \frac{n|x|}{n^2x^2} = \frac{1}{n|x|} < \frac{1}{N_\varepsilon|x|} < \varepsilon$$

whenever $n > N_\varepsilon$, i.e. $\frac{nx}{1+n^2x^2} \to 0$ on $\mathbb{R}$. ■

89. Show that if $f_n(x) = x + \frac{1}{n}$ and $f(x) = x$ for all $x \in \mathbb{R}$, $n \in \mathbb{N}$, then $f_n \rightrightarrows f$ on $\mathbb{R}$ but $f_n^2 \not\rightrightarrows g$ on $\mathbb{R}$ for any function g.

Proof. Let $\varepsilon > 0$. By the Archimedean property, $\exists N_\varepsilon > \frac{1}{\varepsilon}$ (independent of x) s.t.

$$|f_n(x) - f(x)| = \left| x + \frac{1}{n} - x \right| = \frac{1}{n} < \frac{1}{N_\varepsilon} < \varepsilon$$

whenever $n > N_\varepsilon$, i.e. $f_n \rightrightarrows 0$ on $\mathbb{R}$.

Now, $(f_n(x))^2 = x^2 + \frac{2x}{n} + \frac{1}{n^2} \to x^2$ for all $x \in \mathbb{R}$. Hence, $f_n^2 \to g$ on $\mathbb{R}$, where $g(x) = x^2$. If f_n^2 converges uniformly to any function, it will have to do so to g. But let $\varepsilon_0 = 2$ and $x_n = n$. Then

$$\left| (f_n(x_n))^2 - g(x_n) \right| = \left| \frac{2x_n}{n} + \frac{1}{n^2} \right| = 2 + \frac{1}{n^2} \geq 2 = \varepsilon_0$$

for all $n \in \mathbb{N}$ (this is the negation of the definition of uniform convergence). Hence f_n^2 does not converge uniformly on $\mathbb{R}$. ■

90. Let $f_n(x) = \frac{1}{(1+x)^n}$ for $x \in [0,1]$. Denote by f the pointwise limit of f_n on $[0,1]$. Does $f_n \rightrightarrows f$ on $[0,1]$?

Solution. First note that $1 \leq 1 + x$ on $[0, 1]$.

In particular, $\frac{1}{1+x} \leq 1$ on $[0, 1]$. If $x \in (0, 1]$, then $\frac{1}{(1+x)^n} \to 0$, according to one of the examples done in class.

If $x = 0$, $\frac{1}{(1+x)^n} = \frac{1}{1^n} = 1 \to 1$; i.e. $f_n \to f$ on $[0, 1]$, where

$$f(x) = \begin{cases} 0, & x \in (0, 1] \\ 1, & x = 0 \end{cases}.$$

However, $f_n \not\rightrightarrows f$ by theorem 67, since f_n is continuous on $[0, 1]$ for all $n \in \mathbb{N}$, but f is not. ■

91. Let (f_n) be the sequence of functions defined by $f_n(x) = \frac{x^n}{n}$, for $x \in [0, 1]$ and $n \in \mathbb{N}$.

Show that (f_n) converges uniformly to a differentiable function $f : [0, 1] \to \mathbb{R}$, and that the sequence (f'_n) converges pointwise to a function $g : [0, 1] \to \mathbb{R}$, but that $g(1) \neq f'(1)$.

Proof. The sequence $f_n(x) = \frac{x^n}{n} \to f(x) \equiv 0$ on $[0,1]$.

Indeed, let $\varepsilon > 0$. By the Archimedean Property, $\exists N_\varepsilon > \frac{1}{\varepsilon}$ s.t.

$$\left| \frac{x^n}{n} - 0 \right| \leq \frac{|x|^n}{n} \leq \frac{1}{n} < \frac{1}{N_\varepsilon} < \varepsilon$$

whenever $n > N_\varepsilon$. Note that f is differentiable and $f'(x) = 0$ for all $x \in [0,1]$. Furthermore, $f_n'(x) = \frac{nx^{n-1}}{n} = x^{n-1} \to g(x)$ on $[0,1]$, where

$$g(x) = \begin{cases} 0, & x \in [0,1) \\ 1, & x = 1 \end{cases},$$

by one of the examples I did in class. Then $g(1) = 1 \neq 0 = f'(1)$. ■

92. Show that $\displaystyle\lim_{n\to\infty}\int_1^2 e^{-nx^2}\,dx = 0.$

Proof. As $\left(e^{-nx^2}\right)' = -2nxe^{-nx^2} < 0$ on $[1,2]$ for all $n \in \mathbb{N}$, e^{-nx^2} is decreasing on $[1,2]$ for all n, that is

$$e^{-nx^2} \leq e^{-n(1)^2} = e^{-n} \quad \text{for all } n \in \mathbb{N}.$$

Now,

$$f_n(x) = e^{-nx^2} \rightrightarrows f(x) \equiv 0 \quad \text{on } [1,2].$$

Indeed, let $\varepsilon > 0$. By the Archimedean Property, $\exists N_\varepsilon > \ln \frac{1}{\varepsilon}$ (independent of x) s.t.

$$\left|e^{-nx^2} - 0\right| = e^{-nx^2} < e^{-Nx^2} \leq e^{-N} < \varepsilon$$

whenever $n > N_\varepsilon$. Then (and only because of this uniform convergence),

$$\lim_{n\to\infty}\int_1^2 e^{-nx^2}\,dx = \int_1^2 \lim_{n\to\infty} e^{-nx^2}\,dx = \int_1^2 0\,dx = 0,$$

by the Limit Interchange Theorem for Integrals. ■