MAT 2125
Elementary Real Analysis

Exercises – Solutions – Q93-Q100

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93. Answer the following questions about series.

(a) If $\displaystyle\sum_{k=1}^{\infty}(a_k + b_k)$ converges, what about $\displaystyle\sum_{k=1}^{\infty} a_k$ and $\displaystyle\sum_{k=1}^{\infty} b_k$?

(b) If $\displaystyle\sum_{k=1}^{\infty}(a_k + b_k)$ diverges, what about $\displaystyle\sum_{k=1}^{\infty} a_k$ and $\displaystyle\sum_{k=1}^{\infty} b_k$?

(c) If $\displaystyle\sum_{k=1}^{\infty}(a_{2k} + a_{2k-1})$ converges, what about $\displaystyle\sum_{k=1}^{\infty} a_k$?

(d) If $\displaystyle\sum_{k=1}^{\infty} a_k$ converges, what about $\displaystyle\sum_{k=1}^{\infty}(a_{2k} + a_{2k-1})$?

Solution.

(a) They might both diverge. Consider $a_k = -k$ and $b_k = k$. However, if one converges, then so does the other, by the arithmetic of limits/series.

(b) At least one of them diverges because if they both converged, then the series of sums would converge as well (according to a proposition seen in class).

(c) Nothing. Consider $a_{2k} = k$, $a_{2k+1} = -k$, for which $\displaystyle\sum_{k=1}^{\infty} a_k$ diverges, but $a_{2k} = \frac{1}{k^2}$, $a_{2k+1} = 0$, for which $\displaystyle\sum_{k=1}^{\infty} a_k$ converges.

(d) It also converges. The sequence of partial sums of the second series is

$$(a_1 + a_2, a_1 + a_2 + a_3 + a_4, , a_1 + a_2 + a_3 + a_4 + a_5 + a_6, \ldots)$$

is a subsequence of the sequence of partial sums of the first series

$$(a_1, a_1 + a_2, a_1 + a_2 + a_3, a_1 + a_2 + a_3 + a_4, \ldots).$$

If the first series sequence of partial sums converges, so does the subsequence's series. ■

94. Show that

$$\frac{1}{r-1}=\frac{1}{r+1}+\frac{2}{r^2+1}+\frac{4}{r^4+1}+\frac{8}{r^8+1}+\cdots$$

for all $r>1$.

Hint: Note that

$$\frac{1}{\ell-1}-\frac{1}{\ell+1}=\frac{2}{\ell^2-1}.$$

Proof. From the hint, we see that

$$\frac{1}{\ell+1} = \frac{1}{\ell-1} - \frac{2}{\ell^2-1}.$$

Thus, for all $k \in \mathbb{N}$, if $\ell = 2^k$, we have

$$\begin{aligned} \frac{1}{r^{2^k}+1} &= \frac{1}{r^{2^k}-1} - \frac{2}{r^{2^{k+1}}-1} \\ \Longrightarrow \frac{2^k}{r^{2^k}+1} &= \frac{2^k}{r^{2^k}-1} - \frac{2^{k+1}}{r^{2^{k+1}}-1}. \end{aligned}$$

Therefore, we have a telescoping sum

$$\sum_{k=1}^{\infty} \frac{2^k}{r^{2^k}+1} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{2^k}{r^{2^k}+1} = \lim_{n\to\infty} \left(\frac{1}{r-1} - \frac{2^n}{r^{2^n}-1} \right) = \frac{1}{r-1},$$

where the last equality follows from the fact that, for $r > 1$, we have

$$\lim_{m\to\infty} \frac{m}{r^m} = 0.$$

This completes the proof. ■

95. Find the values of p for which the series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^p}$ converges.

Proof. If $p \leq 0$, then $\frac{1}{n^p} \not\to 0$ so the series diverges. In what follows, then, let $p > 0$.

For $k \in \mathbb{N}$, consider the function $f_{k;p} : [1,k] \to \mathbb{R}$ defined by $f_{k;p}(x) = \frac{1}{x^p}$. Since $f'_{k;p}(x) = -\frac{p}{x^{p+1}} < 0$ for all $x \geq 1$, $f_{k;p}$ is strictly decreasing on $[1,k]$. Thus $f_{k;p}$ is Riemann-integrable on $[1,k]$.

Consider the partition $P_k = \{1, 2, \ldots, k, k+1\}$ of $[1, k+1]$. Since $f_{k;p}$ is Riemann-integrable,

$$L(f_{k;p}; P_k) \leq \int_1^{k+1} f_{k;p} \leq U(f_{k;p}; P_k).$$

As $f_{k;p}$ is decreasing on the sub-interval $[\mu, \nu]$, $f_{k;p}$ reaches its maximum at μ and its minimum at ν;

Hence

$$U(f_{k;p};P_k)=\sum_{n=1}^{k} f_{k;p}(n)(n+1-n)=\sum_{n=1}^{k}\frac{1}{n^p},\quad \text{and}$$

$$L(f_{k;p};P_k)=\sum_{n=2}^{k+1} f_{k;p}(n+1)(n+1-n)=\sum_{n=2}^{k+1}\frac{1}{n^p}.$$

But

$$\sum_{n=2}^{k+1}\frac{1}{n^p}=\frac{1}{(k+1)^p}-1+\sum_{n=1}^{k}\frac{1}{n^p}.$$

Thus

$$\frac{1}{(k+1)^p}-1+\sum_{n=1}^{k}\frac{1}{n^p}\leq\int_1^{k+1} f_{k;p}\leq\sum_{n=1}^{k}\frac{1}{n^p}.$$

Write $s_{k;p}$ for the partial sum and note that

$$\int_1^{k+1} f_{k;p} = \int_1^{k+1} \frac{dx}{x^p} = \begin{cases} \ln(k+1), & \text{when } p = 1 \\ \frac{1}{1-p}(k^{1-p} - 1), & \text{when } p \neq 1 \end{cases}$$

If $p = 1$, then $\ln(k+1) \leq s_{k;1}$ for all k. Since the sequence $\{\ln(k+1)\}_k$ is unbounded, so must $\{s_{k;1}\}_k$ be unbounded, which means that the corresponding series cannot converge.

If $p > 1$, then

$$\lim_{k\to\infty} \left(\frac{1}{1-p}(k^{1-p} - 1) + 1 - \frac{1}{(k+1)^p} \right) = \frac{p}{p-1}.$$

Since $s_{k;p}$ is monotone (as every additional $\frac{1}{n^p}$ added to the partial sum is positive) and since $s_{k;p}$ is bounded above by the convergent sequence

$$\left\{\frac{1}{1-p}(k^{1-p}-1)+1-\frac{1}{(k+1)^p}\right\}_k,$$

$s_{k;p}$ is a convergent sequence.

If $p < 1$, then

$$\left\{\frac{1}{1-p}(k^{1-p}-1)\right\}_k$$

is unbounded. As $s_{k;p} \geq \frac{1}{1-p}(k^{1-p}-1)$ for all k, $\{s_{k;p}\}$ is also unbounded, which means that the corresponding series cannot converge.

Thus, the series converges if and only if $p > 1$. ■

96. Which of the following series converge?

(a) $$\sum_{n=1}^{\infty} \frac{n(n+1)}{(n+2)^2}$$

(b) $$\sum_{n=1}^{\infty} \frac{2+\sin^3(n+1)}{2^n+n^2}$$

(c) $$\sum_{n=1}^{\infty} \frac{1}{2^n-1+\cos^2 n^3}$$

(d) $$\sum_{n=1}^{\infty} \frac{n+1}{n^2+1}$$

(e) $$\sum_{n=1}^{\infty} \frac{n+1}{n^3+1}$$

(f) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$

(g) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{5^n}$

(h) $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{3^{1+2n}}$

(i) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{5n+3n^3}{7n^3+2}\right)^n$

Solution. We use the various tests at our disposal.

(a) Since

$$\lim_{n\to\infty} \frac{n(n+1)}{(n+2)^2} = 1 \neq 0,$$

the series diverges .

(b) Since $-1 \leq \sin^3(n+1) \leq 1$, we have

$$0 \leq \frac{2+\sin^3(n+1)}{2^n+n^2} \leq \frac{1}{2^n+n^2} \leq \frac{1}{2^n}.$$

Thus the given series converges by comparison with the geometric series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n}$.

(c) If a_n denotes the n-th term of the series, we have

$$\frac{a_{n+1}}{a_n} = \frac{2^n - 1 + \cos^2 n^3}{2^{n+1} - 1 + \cos^2(n+1)^3} \to \frac{1}{2} < 1.$$

Thus the series converges by the ratio test.

(d) We have

$$\frac{n+1}{n^2+1} \geq \frac{n}{2n^2} = \frac{1}{2n}.$$

Thus the series diverges by comparison with the harmonic series.

(e) We have

$$0 \leq \frac{n+1}{n^3+1} \leq \frac{2n}{n^3} = \frac{2}{n^2}.$$

Thus the series converges by comparison with $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n^2}$.

(f) For $n \geq 2$, we have

$$0 \leq \frac{n!}{n^n} = \frac{1}{n} \cdot \frac{2}{n} \cdot \frac{3 \cdot 4 \cdots n}{n^{n-2}} \leq \frac{2}{n^2}.$$

Thus the series converges by comparison with $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n^2}$.

(g) If a_n denotes the n-th term in the series, we have

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{5^{n+1}} \frac{5^n}{n!} = \frac{n+1}{5} \to \infty.$$

Thus the series diverges by the ratio test.

(h) We have

$$\left(\frac{n^n}{3^{1+2n}}\right)^{1/n} = \frac{n}{3^{2+1/n}} \to \infty.$$

Thus the series diverges by the root test.

(i) We have

$$\left(\left(\frac{5n+3n^3}{7n^3+2}\right)^n\right)^{1/n} = \frac{5n+3n^3}{7n^3+2} \to \frac{3}{7} < 1.$$

Thus the series converges by the root test. ■

97. Give an example of a power series $\sum_{k=0}^{\infty} a_k x^k$ with interval of convergence exactly $[-\sqrt{2}, \sqrt{2})$.

Proof. Consider the series
$$\sum_{k=1}^{\infty}\frac{x^k}{k}.$$
We have
$$\limsup_{k\to\infty}\sqrt[k]{\frac{|x|^k}{k}}=\limsup_{k\to\infty}\frac{|x|}{\sqrt[k]{k}}=|x|.$$
Therefore, by the root test, the series converges when $|x|<1$ and diverges for $|x|>1$.

For $x=1$, the series is the harmonic series, which diverges. For $x=-1$, it is the alternating harmonic series, which converges.

Thus, the series converges precisely on the interval $[-1,1)$.

Now, replace x by $x/\sqrt{2}$. The corresponding power series is thus

$$\sum_{k=0}^{\infty} \frac{1}{\sqrt{2}^k k} x^k.$$

We have

$$\limsup_{k\to\infty} \sqrt[k]{\frac{|x|^k}{\sqrt{2}^k k}} = \limsup_{k\to\infty} \frac{|x|}{\sqrt{2}\sqrt[k]{k}} = \frac{|x|}{\sqrt{2}}.$$

The series converges on $\frac{|x|}{\sqrt{2}} < 1$ and diverges on $\frac{|x|}{\sqrt{2}} > 1$. For $x = \sqrt{2}$, the series is the harmonic series, which diverges. For $x = -\sqrt{2}$, it is the alternating harmonic series, which converges.

Thus, the series converges precisely on the interval $[-\sqrt{2}, \sqrt{2})$. ■

98. Find the values of x for which the following series converge:

(a) $\displaystyle\sum_{n=1}^{\infty}(nx)^n$;

(b) $\displaystyle\sum_{n=1}^{\infty}x^n$;

(c) $\displaystyle\sum_{n=1}^{\infty}\frac{x^n}{n^2}$;

(d) $\displaystyle\sum_{n=1}^{\infty}\frac{x^n}{n!}$.

Solution.

(a) The series diverges whenever $x \neq 0$ since the terms $(nx)^n$ do not tend to zero when $n \to \infty$. (For large enough n, we have $n|x| \geq 1$.) Thus, this power series converges *only* at its center.

(b) The geometric series converges precisely on the interval $(-1, 1)$, and the series takes on the value $\frac{1}{1-x}$ there.

(c) For $|x| \leq 1$, we have
$$\left|\frac{x^n}{n^2}\right| \leq \frac{1}{n^2},$$
and thus the series converges for these values of x. If $|x| > 1$, the terms $|x^n/n^2| \to \infty$, and so the series diverges. Hence the series converges precisely on the interval $[-1, 1]$.

(d) Let $x \in \mathbb{R}$. Using the ratio test we have

$$\frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} = \frac{x}{n+1} \to 0.$$

Thus the series converges for all $x \in \mathbb{R}$ (and takes on the value e^x). ■

99. If the power series $\sum_{k=0}^{\infty} a_k x^k$ has radius of convergence R, what is the radius of convergence of the series $\sum_{k=0}^{\infty} a_k x^{2k}$?

Solution. The new series can be written as $\sum_{k=0}^{\infty} b_k x^k$, where $b_k = a_{k/2}$ if k is even and $b_k = 0$ if k is odd. Thus

$$\limsup_{k\to\infty} \sqrt[k]{|b_k|} = \lim_{k\to\infty} \sqrt[k]{|a_{k/2}|} = \lim_{k\to\infty} \sqrt[2k]{|a_k|} = \lim_{k\to\infty} \left(\sqrt[k]{|a_k|}\right)^{1/2}$$

$$= \left(\lim_{k\to\infty} \sqrt[k]{|a_k|}\right)^{1/2} = R^{1/2}.$$

Therefore, the radius of convergence of the new series is $\sqrt{R}$. ■

100. Obtain power series expansions for the following functions.

(a) $\dfrac{x}{1+x^2}$;

(b) $\dfrac{x}{(1+x^2)^2}$;

(c) $\dfrac{x}{1+x^3}$;

(d) $\dfrac{x^2}{1+x^3}$;

(e) $f(x) = \displaystyle\int_0^1 \frac{1-e^{-sx}}{s}\,ds$, about $x=0$.

Solution.

(a) Since

$$\frac{1}{1-x}=\sum_{k=0}^{\infty}x^k,$$

we have

$$\frac{x}{1+x^2}=x\sum_{k=0}^{\infty}(-x^2)^k=\sum_{k=0}^{\infty}(-1)^k x^{2k+1}.$$

(b) We know that, for $x\in(-1,1)$, $\displaystyle\frac{1}{1-x}=\sum_{k=1}^{\infty}x^k$.

For any $-1<a<b<1$, the series $\displaystyle\sum_{k=1}^{\infty}kx^{k-1}$ converges uniformly on $[a,b]$.

Indeed, let $c = \max\{|a|, |b|\} < 1$. Then, for all $x \in [a, b]$, we have

$$|kx^{k-1}| \leq kc^{k-1}.$$

Now,

$$\frac{(k+1)c^k}{kc^{k-1}} = \frac{k+1}{k}c \to c \quad \text{as } k \to \infty.$$

Since $c < 1$, the ratio test tells us that $\sum_{k=1}^{\infty} kc^{k-1}$ converges.

Thus, $\sum_{k=1}^{\infty} kx^{k-1}$ converges uniformly by the Weierstrass M-test.

Consequently, we have

$$\sum_{k=1}^{\infty} kx^{k-1} = \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2},$$

and so for any $x \in [a,b] \subseteq (-1,1)$:

$$\frac{x}{(1+x^2)^2} = x\sum_{k=1}^{\infty} k(-x^2)^{k-1} = \sum_{k=1}^{\infty}(-1)^{k-1}kx^{2k-1}.$$

(c) Using the geometric series, we have

$$\frac{x}{1+x^3} = x\sum_{k=0}^{\infty}(-x^3)^k = \sum_{k=0}^{\infty}(-1)^k x^{3k+1}.$$

(d) Using the geometric series, we have

$$\frac{x^2}{1+x^3} = x^2\sum_{k=0}^{\infty}(-x^3)^k = \sum_{k=0}^{\infty}(-1)^k x^{3k+2}.$$

(e) Since

$$e^x = \sum_{k=0}^{\infty}\frac{x^k}{k!},$$

we have

$$\frac{1-e^{-sx}}{s} = -\frac{1}{s}\sum_{k=1}^{\infty}\frac{(-sx)^k}{k!} = \sum_{k=1}^{\infty}(-1)^{k+1}\frac{s^{k-1}x^k}{k!}.$$

This series converges absolutely for all s and all x (use the ratio test or compare it to the series for e^x). Therefore, viewing it as a power series in s (for some fixed x), its interval of convergence is ∞, and its centre is 0. Thus the series can be integrated term by term:

$$\begin{aligned}
\int_0^1 \frac{1-e^{-sx}}{s}\,ds &= \int_0^1 \sum_{k=1}^{\infty}(-1)^{k+1}\frac{s^{k-1}x^k}{k!}\,ds \\
&= \sum_{k=1}^{\infty}(-1)^{k+1}\left(\int_0^1 s^{k-1}\,ds\right)\frac{x^k}{k!} \\
&= \sum_{k=1}^{\infty}(-1)^{k+1}\left[\frac{s^k}{k}\right]_{s=0}^{s=1}\frac{x^k}{(k!)} = \sum_{k=1}^{\infty}(-1)^{k+1}\frac{x^k}{k(k!)}.
\end{aligned}$$

This completes the exercises for the course. ■