MAT 2125
Elementary Real Analysis

Chapter 3
Sequences

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Overview

A large chunk of analysis concerns itself with problems of convergence. In this chapter, we

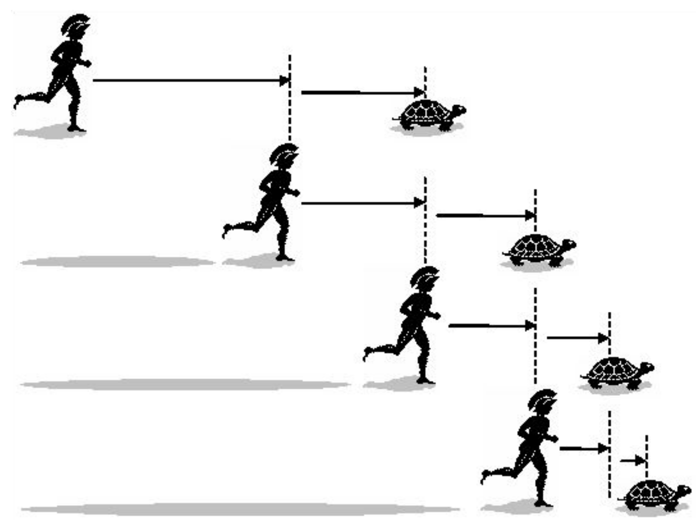
- introduce sequences and limits,
- provide results that help to compute such limits, and
- identify situations when the limit can be shown to exist without having to compute it.

Outline

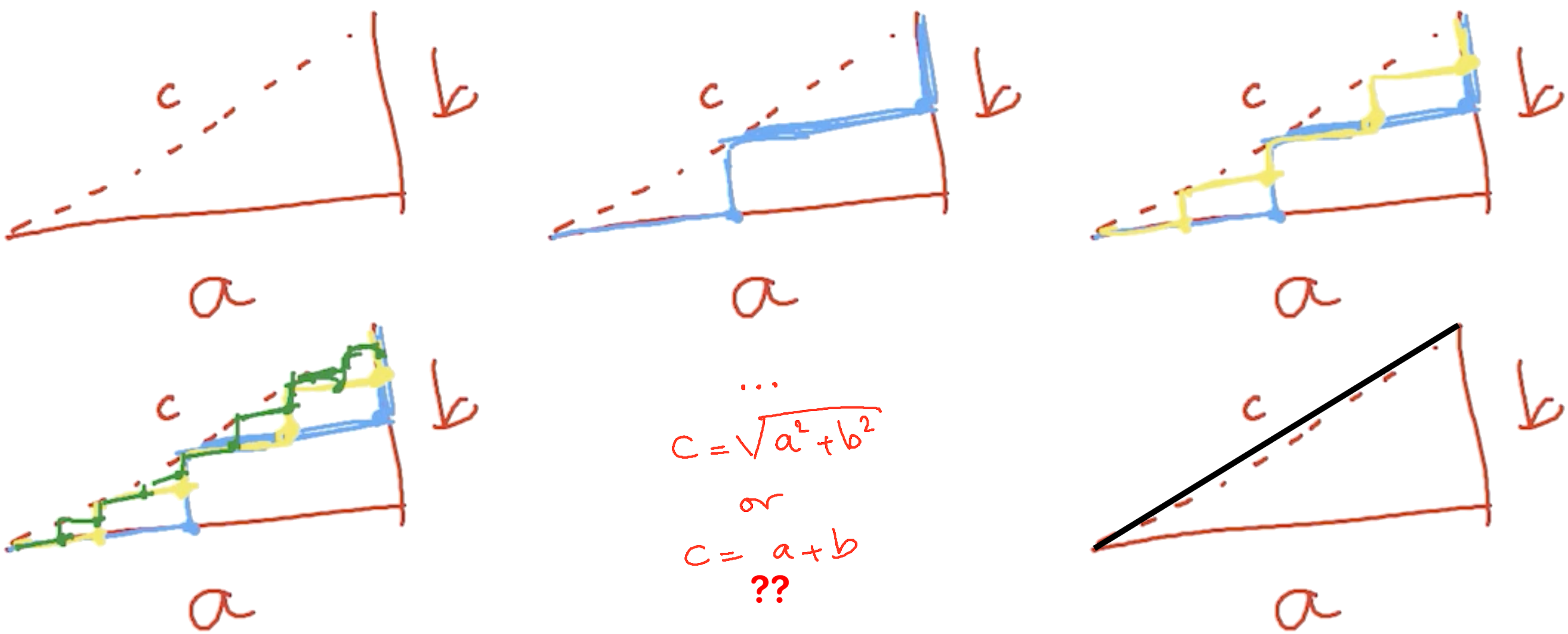
3.1 – Infinity vs. Intuition

When dealing with infinity, our intuition sometimes falters.

Example: (Zeno's Paradox)
Achilles pursues a turtle. When he reaches her starting point, she has moved a certain distance. When he crosses that distance, she has moved yet another distance, and so forth. Achilles is always trailing the turtle, so he cannot catch her. Is this what happens in reality?



Example: (Anti-Pythagorean Theorem)
Consider a right-angle triangle with base a, height b, and hypotenuse c. We can build staircase structures that each have the same constant length as $a+b$, while increasing the number of stairs (see image below).



Example: (Infinite Sum I)
Let $S = 1 + (-1) + 1 + (-1) + \cdots$. Then

$$S = (1 + (-1)) + (1 + (-1)) + \cdots = 0 + 0 + \cdots = 0$$

$$S = 1 - (1 + (-1) + 1 + (-1) + \cdots) = 1 + S \implies S = 1/2$$

$$S = 1 + ((-1) + 1) + ((-1) + 1) + \cdots = 1 + 0 + 0 + \cdots = 1$$

Therefore $0 = \frac{1}{2} = 1$. Does this make sense?

Example: (Infinite Sum II)
Let $S = 1 + 2 + 4 + 8 + \cdots$. Then

$$S = 1 + 2(1 + 2 + 4 = 8 + \cdots) = 1 + 2S \implies S = -1.$$

Can a sum of positive terms yield a negative result?

3.2 – Limit of a Sequence

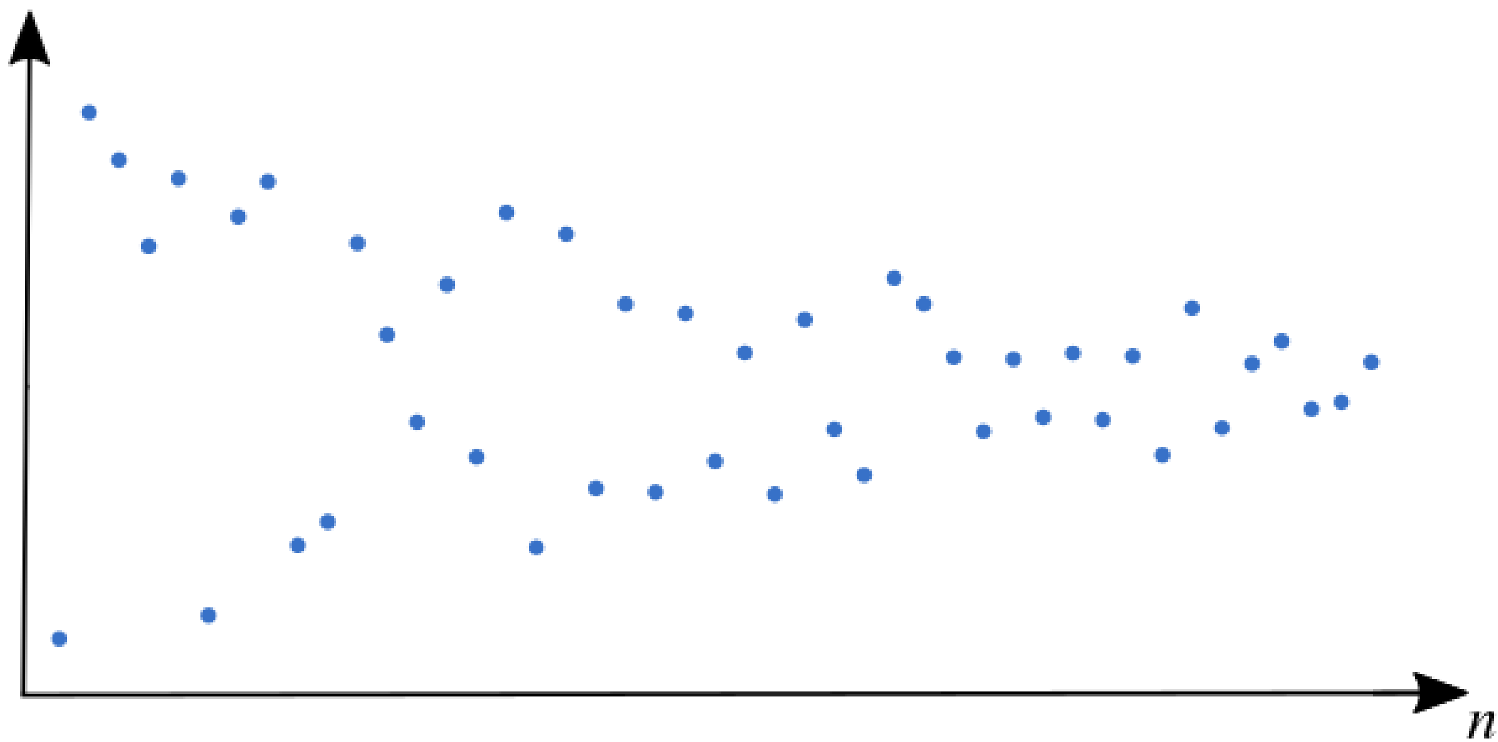
A **sequence** of real numbers is a function $X : \mathbb{N} \to \mathbb{R}$ defined by $X(n) = a_n$, where $a_n \in \mathbb{R}$. We denote the sequence X by $(a_n)_{n\in\mathbb{N}}$ or simply by (a_n).

Examples:

1.

2.

In general, we let $\mathbb{N}$ stand for whatever countable subset of $\mathbb{N}$ is required for the definition of the sequence to make sense.



A sequence as a function on $\mathbb{N}$.

The notation used for sequences varies from one resource to the next.

We will mostly use round brackets:

$$(a_n) \text{ where } a_n = \frac{1}{n^2}, \quad \left(\frac{1}{n^2}\right), \quad \left(1, \frac{1}{4}, \frac{1}{9}, \ldots\right)$$

all denote the same sequence.

A sequence is an ordered set of **terms** a_n, that is, a set of **indexed values**. The set of all values taken by the sequence (a_n) is called the **range** of (a_n) and we denote it by $\{a_n\}$.

A sequence and its range are two different notions.

Examples:

1.

2.

Certain sequences are defined with the help of a **recurrence relation**: the first few terms are given, and the subsequent terms are computed using the preceding terms.

Example:

We now examine in detail the sequence $(x_n) = (\frac{1}{2n}) = (\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{78}, \ldots)$.

As the index n increases, the values of x_n **approach** 0. But what does this mean, mathematically?

A sequence (x_n) of real numbers is said to **converge** to a **limit** $L \in \mathbb{R}$, denoted by

$$x_n \to L \quad \text{or} \quad \lim_{n\to\infty} x_n = L$$

if

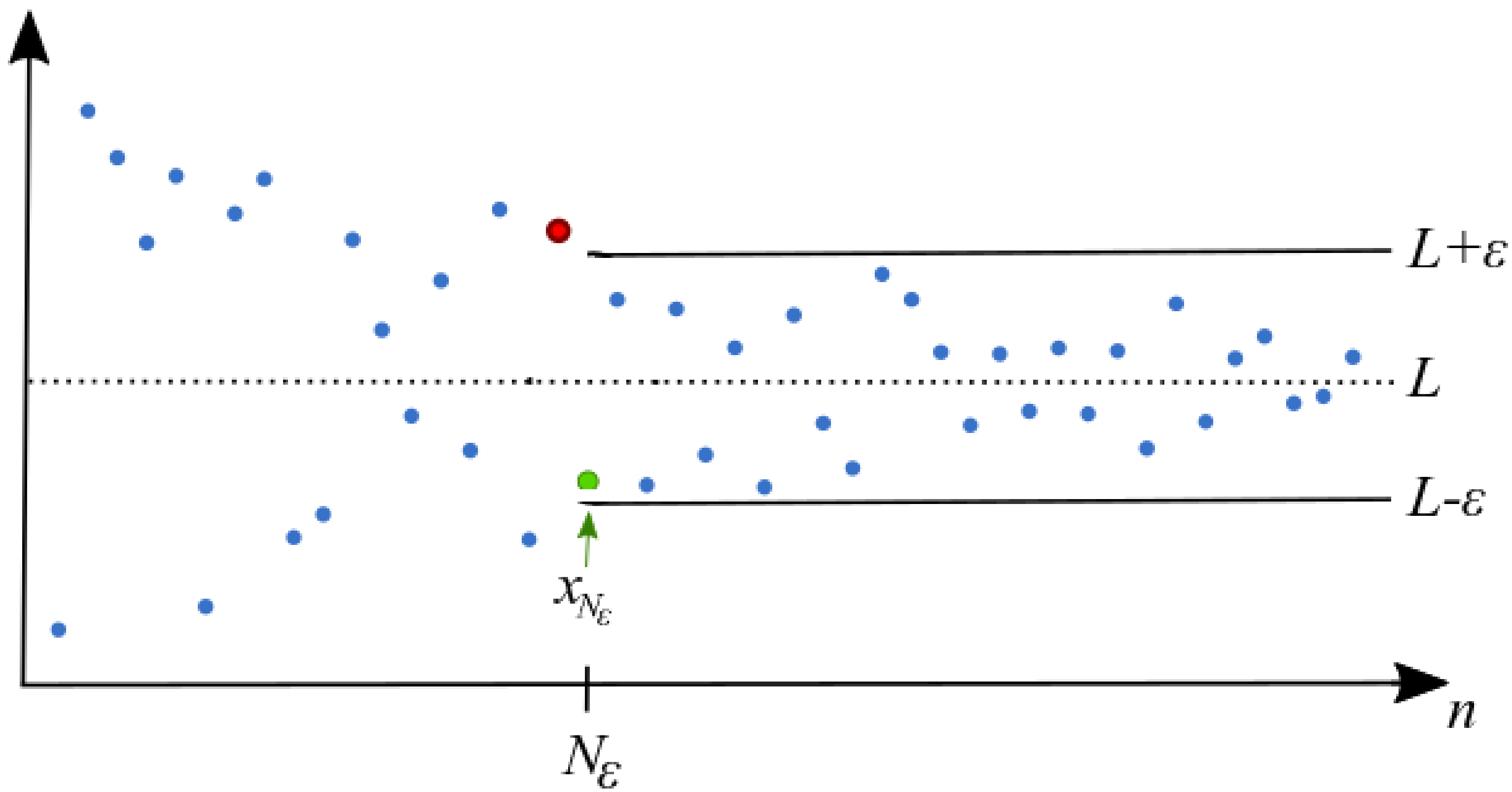
$$\forall \varepsilon > 0,\ \exists N_\varepsilon \in \mathbb{N} \text{ such that } n > N_\varepsilon \implies |x_n - L| < \varepsilon.$$

A sequence (x_n) which does not converge to a limit is said to be **divergent**:

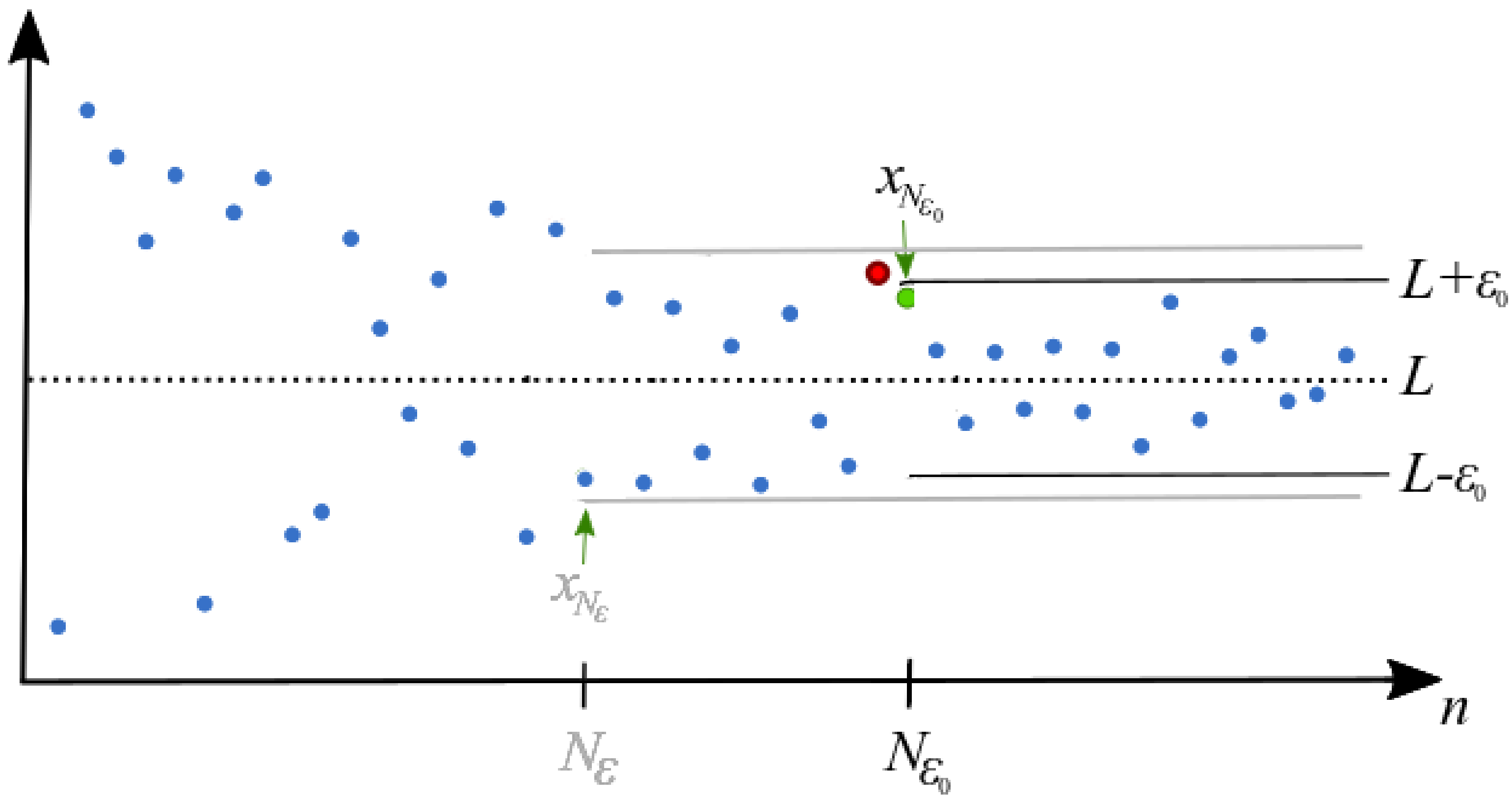
$$\forall L \in \mathbb{R},\ \exists \varepsilon_L > 0,\ \forall N \in \mathbb{N},\ \exists n_N > N \text{ such that } |x_{n_N} - L| \geq \varepsilon_L.$$

There is only one way for a sequence to converge: its values are getting closer and closer to the limit. But there is more than one way for a sequence to diverge.

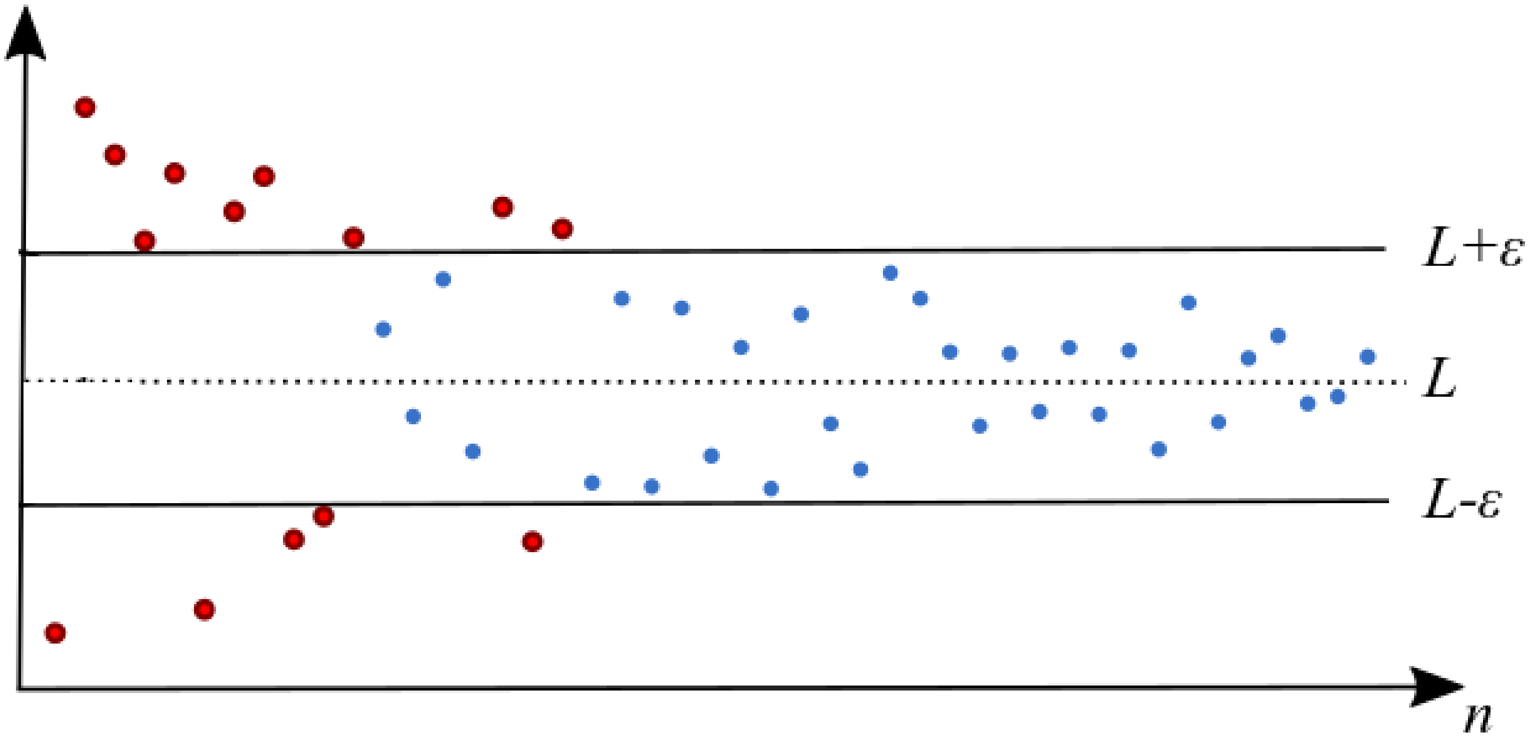
Can you think of some?



$L+\varepsilon$
L
$L-\varepsilon$
x_{N_ε}
N_ε
n



$x_{N_{\varepsilon_0}}$
$L+\varepsilon_0$
L
$L-\varepsilon_0$
x_{N_ε}
n
N_ε
N_{ε_0}


$L+\varepsilon$
L
L-ε
n

Examples:

1. Show that $\frac{1}{n} \to 0$.

   Proof.

   $\blacksquare$

2. Show that $\frac{n+1}{n^2+1} \to 0$.

Proof.

■

3. Show that $\frac{4-2n-3n^2}{2n^2+n} \to -\frac{3}{2}$.

Proof.

■

4. Show that (n) is divergent.

Proof.

■

The main benefit of the formal definition of the limit of a sequence is that it does not call on infinity: we write $n \to \infty$, but that is a merely a notation convenience.

On the flip side, the formal definition has 2 major inconveniences:

1. it cannot be used to **determine the limit** of a convergent sequence – it can only be used to verify that a given candidate is (or is not) a limit of a sequence;

2. it can seem artificial to some extent, especially upon a first encounter.

The goal is simple: we must **determine a threshold** N_ε that does the trick. This often requires **backtracking** from the end of the string of inequalities rather than to proceed directly from "Let $\varepsilon > 0$".

We have been careful to refer to "a" limit when the sequence converges, but we should really be talking about "the" limit in such cases.

Theorem 12. (UNIQUE LIMIT) *A convergent sequence* (x_n) *of real numbers has exactly one limit.*

Proof.

■

A sequence $(x_n) \subseteq \mathbb{R}$ is **bounded** by $M > 0$ if $|x_n| \leq M$ for all $n \in \mathbb{N}$.

Theorem 13. *Any convergent sequence (x_n) of real numbers is bounded.*

Proof.

$\blacksquare$

We can prove theorems **directly**, as in Theorem 13, by induction, as in Bernouilli's Inequality, or by **contradiction**, as in the Archimedean Property.

The **contrapositive** of a statement $P \implies Q$ is $\neg Q \implies \neg P$. These two statements are **logically equivalent** to one another; it may be that it is easier to demonstrate the contrapositive than the original statement.

The **converse** of a statement $P \implies Q$ is $Q \implies P$. There is no general link between a statement and its converse: sometimes they are both true, sometimes they are both false, sometimes only of them is true.

Example:

3.3 – Operations on Sequences

The following result removes the need to use the formal definition.

Theorem 14. (Operations on Convergent Sequences)
Let $(x_n), (y_n)$ *be convergent, with* $x_n \to x$ *and* $y_n \to y$. *Let* $c \in \mathbb{R}$. *Then*

1. $|x_n| \to |x|$;
2. $(x_n + y_n) \to (x + y)$;
3. $x_n y_n \to xy$ *and* $cx_n \to cx$;
4. $\frac{x_n}{y_n} \to \frac{x}{y}$, *if* $y_n, y \neq 0$ *for all* n.

Proof. We show each part using the definition of the limit of a sequence.

1.

2.

(1)

3.

(2)

4.

(3)

(4)

■

Can the limit of a sequence whose terms are all near 2 be -19? 0? 1? 2?

Theorem 15. (Comparison Theorem for Sequences)
Let $(x_n), (y_n)$ be convergent sequences of real numbers with $x_n \to x$, $y_n \to y$, and $x_n \leq y_n$ $\forall n \in \mathbb{N}$. Then $x \leq y$.

Proof.

$\blacksquare$

⚠ The "$\leq$"s in the statement of Theorem 15 cannot be replaced by "$<$"s throughout. For instance, if $(x_n) = (\frac{1}{n+1})$ and $(y_n) = (\frac{1}{n})$, then $x_n < y_n$ for all $n \in \mathbb{N}$, but $x_n \to x = 0$, $y_n \to y = 0$, and $0 = x \not< y = 0$.

Theorem 16. (Squeeze Theorem for Sequences)
Let $(x_n), (y_n), (z_n) \subseteq \mathbb{R}$ *be such that* $x_n, z_n \to \alpha$ *and* $x_n \leq y_n \leq z_n$, $\forall n \in \mathbb{N}$. *Then* $y_n \to \alpha$.

Proof.

■

We can use these various results to compute the following limits.

Examples:

1. Compute $\lim_{n\to\infty} \frac{3n+1}{n}$, if the limit exists.

   Solution.

   ■

2. Compute $\lim_{n\to\infty} \frac{\sin(n^2+212)}{n}$, if the limit exists.

Solution.

$\blacksquare$

3. Compute $\lim\limits_{n\to\infty} \dfrac{2n-1}{n+7}$, if the limit exists.

Solution.

■

4. Let (x_n) be such that $|x_n| \to 0$. Show that $x_n \to 0$.

Proof.

■

Note, however that if $|x_n| \to \alpha \neq 0$, we cannot necessarily conclude that $x_n \to \alpha$. Consider, for instance, the sequence $(x_n) = (-1)^n$.

5. Let $|q| < 1$. Compute $\lim\limits_{n\to\infty} q^n$, if the limit exists.

Proof.

6. Let $|q| < 1$. Compute $\lim\limits_{n\to\infty} nq^n$, if the limit exists.

Solution.

7. Show that $\sqrt[n]{n} \to 1$.

Proof.

$\blacksquare$

8. Compute $\lim_{n\to\infty} \dfrac{n!}{n^n}$, if the limit exists.

Solution.

∎

9. Let $a > 0$. Compute $\lim_{n\to\infty} a^{1/n}$, if the limit exists.

Solution.

∎

10. Compute $\lim_{n\to\infty} \sqrt[n]{3^n + 5^n}$, if the limit exists.

Solution.

$\blacksquare$

Theorem 17. *Let* $y_n \to y$*. If* $y_n \geq 0\ \forall n \in \mathbb{N}$*, then* $\sqrt{y_n} \to \sqrt{y}$*.*

Proof.

$\blacksquare$

3.4 – Bounded Monotone Convergence Theorem

A sequence (x_n) is **increasing** if

$$x_1 \leq x_2 \leq \cdots x_n \leq x_{n+1} \leq \cdots, \quad \forall n \in \mathbb{N}$$

and it is **decreasing** if

$$x_1 \geq x_2 \geq \cdots \geq x_n \geq x_{n+1} \cdots, \quad \forall n \in \mathbb{N}.$$

If (x_n) is either increasing or decreasing, we say that it is **monotone**. If it is both increasing and decreasing, it is **constant**.

When the inequalities are strict, then the sequence is **strictly increasing** or **strictly decreasing**, depending, and thus **strictly monotone**.

Theorem 18. (Bounded Monotone Convergence)
Let (x_n) *be an increasing sequence, bounded above. Then* (x_n) *converges to* $\sup\{x_n \mid n \in \mathbb{N}\}$.

Proof.

$\blacksquare$

A similar result holds for decreasing sequences that are bounded below.

Examples:

- Does the sequence $(x_n) = (1 - \frac{1}{n})$ converge? If so, what is its limit?

  Solution.

- Let (x_n) be defined by $x_n = \sqrt{2x_{n-1}}$ when $n \geq 2$, with $x_1 = 1$. Does (x_n) converge? If so, to what limit?

  Solution.

■

3.5 – Bolzano-Weierstrass Theorem

The main result of this section is a corner stone of analysis, concerning bounded sequences and their subsequences.

Let $(x_n) \subseteq \mathbb{R}$ be a sequence and $n_1 < n_2 < \cdots$ be an increasing string of positive integers. The sequence $(x_{n_k})_k = (x_{n_1}, x_{n_2}, \ldots)$ is a **subsequence** of (x_n), denoted by $(x_{n_k}) \subseteq (x_n)$. Note that $n_k \geq k$ for all $k \in \mathbb{N}$.

Examples:

-

-

-

-

Theorem 19. *Let* $x_n \to x$. *If* $(x_{n_k}) \subseteq (x_n)$, *then* $x_{n_k} \to x$ *as well.*

Proof.

■

The converse of this theorem is false: can you find a divergent sqeuence with convergent subsequences?

The next result is surprising, deep and useful.

Theorem 20. (BOLZANO-WEIERSTRASS)
If $(x_n) \subseteq \mathbb{R}$ is bounded, it has (at least) one convergent subsequence.

Proof.

■

We have mentioned that a sequence (x_n) which diverges is one for which

$$\forall L \in \mathbb{R},\ \exists \varepsilon_L > 0,\ \forall N \in \mathbb{N},\ \exists n_N > N \text{ such that } |x_{n_N} - L| \geq \varepsilon_L.$$

If (x_n) does not converge to L, it is easy to construct a subsequence (x_{n_k}) that also fails to converge to L:

- let $n_1 \in \mathbb{N}$ be such that $n_1 \geq 1$ and $|x_{n_1} - L| \geq \varepsilon_L$;
- let $n_2 \in \mathbb{N}$ be such that $n_2 \geq n_1$ and $|x_{n_2} - L| \geq \varepsilon_L$;
- etc.

Note that there might be some subsequences of (x_n) that do converge to some L, however: $x_n = (-1)^n$ diverges, but $x_{2n} = (-1)^{2n} = 1 \to 1$.

Theorem 21. *Let $(x_n) \subseteq \mathbb{R}$ be a bounded sequence such that every one of its proper converging subsequence converges to the same $x \in \mathbb{R}$. Then $x_n \to x$.*

Proof.

$\blacksquare$

3.6 – Cauchy Sequences

A main challenge with the definition of a limit of a sequence is that we need to know what the limit is **before** we can show what it is, in which case we do not need to show what it is...

A sequence (x_n) is a **Cauchy sequence** if

$$\forall \varepsilon > 0,\ \exists N_\varepsilon \in \mathbb{N} \text{ such that } m, n > N_\varepsilon \implies |x_m - x_n| < \varepsilon.$$

Incidentally, (x_n) is not a Cauchy sequence if

$$\exists \varepsilon_0 > 0,\ \forall N \in \mathbb{N},\ \exists m_N, n_N > N \text{ such that } |x_{m_N} - x_{n_N}| \geq \varepsilon_0.$$

Examples:

1. Show that $(x_n) = (\frac{1}{n})$ is a Cauchy sequence.

   Proof.

   $\blacksquare$

2. Show that $(x_n) = (1 + \frac{1}{2} + \cdots + \frac{1}{n})$ is not a Cauchy sequence.

Proof.

■

In essence, a Cauchy sequence is a sequence for which the terms can get as close to one another as one wishes, after a threshold (which depends on the desired distance).

The next result shows that Cauchy sequences behave like convergent sequences in $\mathbb{R}$ – we will soon see that the similarity is in fact not pure happenstance.

Theorem 22. *If (x_n) is a Cauchy sequence, then it is bounded.*

Proof.

■

We could also show that the sum of two Cauchy sequences is a Cauchy sequence, that every bounded Cauchy sequence admits at least one convergent subsequence, and so forth.

Cauchy sequences in $\mathbb{R}$ behave like convergent sequences in $\mathbb{R}$ because ...

Theorem 23. *A sequence of real numbers is convergent if and only if it is a Cauchy sequence.*

Proof.

■

Examples:

1.

2. Compute the limit of the sequence defined by $x_n = \frac{1}{2}(x_{n-2} + x_{n-1})$, $n > 2$, with $x_1 = 1$ and $x_2 = 2$.

   Solution.

■

Cauchy sequences illustrate the fundamental difference between $\mathbb{R}$ and $\mathbb{Q}$. A sequence is Cauchy if the points of the sequence "accumulate" on top of one another. In $\mathbb{R}$, every Cauchy sequence is convergent, and *vice-versa*.

In $\mathbb{Q}$, the converging sequences are Cauchy, but there are Cauchy sequences that do not converge: it is possible that the points of such a sequence "accumulate" around of the (uncountably infinitely) many holes of $\mathbb{Q}$.

For instance, the sequence $(1, 1.4, 1.41, 1.414, \ldots)$ is Cauchy in $\mathbb{Q}$, but does not converge in $\mathbb{Q}$.

This leads to one of the ways of building $\mathbb{R}$: we take all Cauchy sequences in $\mathbb{Q}$ and add whatever point the sequences "accoumulates" around to $\mathbb{R}$ (there is more to it than that, but that is the main idea). In this example, we would get to add $\sqrt{2}$ to $\mathbb{R}$.

3.7 – Exercises

1. The first few terms of a sequence (x_n) are given below. Assuming that the "natural pattern" indicated by these terms persists, give a formula for the nth term x_n.
   (a) $(5, 7, 9, 11, \ldots)$;

   (b) $\left(\frac{1}{2}, -\frac{1}{4}, \frac{1}{8}, -\frac{1}{16}, \ldots\right)$;

   (c) $\left(\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots\right)$;

   (d) $(1, 4, 9, 16, \ldots)$.
2. Use the definition of the limit of a sequence to establish the following limits.
   (a) $\lim_{n\to\infty} \left(\frac{1}{n^2+1}\right) = 0$;

   (b) $\lim_{n\to\infty} \left(\frac{2n}{n+1}\right) = 2$;

(c) $\lim_{n\to\infty}\left(\frac{3n+1}{2n+5}\right)=\frac{3}{2}$, and

(d) $\lim_{n\to\infty}\left(\frac{n^2-1}{2n^2+3}\right)=\frac{1}{2}$.

3. Show that

(a) $\lim_{n\to\infty}\left(\frac{1}{\sqrt{n+7}}\right)=0$;

(b) $\lim_{n\to\infty}\left(\frac{2n}{n+2}\right)=2$;

(c) $\lim_{n\to\infty}\left(\frac{\sqrt{n}}{n+1}\right)=0$, and

(d) $\lim_{n\to\infty}\left(\frac{(-1)^n n}{n^2+1}\right)=0$.

4. Show that $\lim_{n\to\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)=0$.

5. Find the limit of the following sequences:
   (a) $\lim_{n\to\infty}\left(\left(2+\frac{1}{n}\right)^2\right)$;
   (b) $\lim_{n\to\infty}\left(\frac{(-1)^n}{n+2}\right)$;
   (c) $\lim_{n\to\infty}\left(\frac{\sqrt{n}-1}{\sqrt{n}+1}\right)$, and
   (d) $\lim_{n\to\infty}\left(\frac{n+1}{n\sqrt{n}}\right)$.
6. Let $y_n=\sqrt{n+1}-\sqrt{n}$. Show that (y_n) and $(\sqrt{n}y_n)$ converge.
7. Let (x_n) be a sequence of positive real numbers such that $x_n^{1/n}\to L<1$. Show $\exists r\in(0,1)$ such that $0<x_n<r^n$ for all sufficiently large $n\in\mathbb{N}$. Use this result to show that $x_n\to 0$.
8. Give an example of a convergent (resp. divergent) sequence (x_n) of positive real numbers with $x_n^{1/n}\to 1$.
9. Let $x_1=1$, $x_{n+1}=\sqrt{2+x_n}$ for $n\in\mathbb{N}$. Show that (x_n) converges; find the limit.

10. Let $x_n = \displaystyle\sum_{k=1}^{n} \frac{1}{k^2}$ for all $n \in \mathbb{N}$. Show that (x_n) is increasing and bounded above.
11. Show that $c^{1/n} \to 1$ if $0 < c < 1$.
12. Let (x_n) be a bounded sequence.
    For each $n \in \mathbb{N}$, let $s_n = \sup\{x_k : k \geq n\}$. If $S = \inf\{s_n\}$, show that there is a subsequence of (x_n) that converges to S.
13. Suppose that $x_n \geq 0$ for all $n \in \mathbb{N}$ and that $((-1)^n x_n)$ converges. Show that (x_n) converges.
14. Show that if (x_n) is unbounded, there exists a subsequence (x_{n_k}) with $1/x_{n_k} \to 0$.
15. If $x_n = \frac{(-1)^n}{n}$, find the convergent subsequence in the proof of the Bolzano-Weierstrass theorem, with $I_1 = [-1, 1]$.
16. Show directly that a bounded increasing sequence is a Cauchy sequence.
17. If $0 < r < 1$ and $|x_{n+1} - x_n| < r^n$ for all $n \in \mathbb{N}$, show that (x_n) is Cauchy.
18. If $x_1 < x_2$ and $x_n = \frac{1}{2}(x_{n-1} + x_{n-2})$ for all $n \in \mathbb{N}$, show that (x_n) is convergent and compute its limit.
19. Suppose that (a_n) is a bounded sequence and $b_n \to 0$. Show that $a_n b_n \to 0$.

20. Consider the sequence given by the recursion $a_{n+1} = \frac{1}{2}(a_n + a_n^{-1})$, with some initial condition $a_1 \in (-\infty, 0) \cup (0, \infty)$. Find and prove the limit, if it exists.
21. Let (a_n) be a sequence with no convergent subsequences. Show that $|a_n| \to \infty$.
22. We define the **limit inferior** and the **limit superior** of a sequence as follows:

$$\liminf_{n\to\infty} a_n = \lim_{n\to\infty} \inf\{a_k \mid k \geq n\}$$

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} \sup\{a_k \mid k \geq n\}.$$

    Let (a_n) be bounded. Show that $\liminf_{n\to\infty} a_n$ and $\limsup_{n\to\infty} a_n$ exist and are in $\mathbb{R}$.
23. Let (a_n) be unbounded. Show that $\liminf_{n\to\infty} a_n = -\infty$ or $\limsup_{n\to\infty} a_n = \infty$.
24. Let (a_n), (b_n) be two sequences. Show that

$$\liminf_{n\to\infty} a_n + \limsup_{n\to\infty} b_n \leq \limsup_{n\to\infty}(a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n.$$

    Furthermore, find a pair of sequences for which the second inequality is strict.

Solutions

1. **Proof.**

2. **Proof.**

3. **Proof.**

4. **Proof.**

5. **Proof.**

6. **Proof.**

7. **Proof.**

8. **Proof.**

9. **Proof.**

10. **Proof.**

11. **Proof.**

12. **Proof.**

13. **Proof.**

14. **Proof.**

15. **Proof.**

16. **Proof.**

17. **Proof.**

18. **Proof.**

19. **Proof.**

$\blacksquare$

20. **Proof.**

■

21. **Proof.**

■

22. **Proof.**

■

23. **Proof.**

$\blacksquare$

24. **Proof.**

■