MAT 2125
Elementary Real Analysis

Chapter 5
Differential and Integral Calculus

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Winter 2022

Overview

We have spent a fair amount of time and energy on concepts like the limit, continuity, and uniform continuity, with the goal of making differential and integral calculus sound.

In this chapter, we

- introduce the concepts of differentiability and Riemann-integrability for functions, and
- prove a number of useful calculus results.

Outline

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5.3 – Exercises (p.101)

5.1 – Differentiation

Let $I \subseteq \mathbb{R}$ be an interval, $f : I \to \mathbb{R}$, and $c \in I$. The real number L is the **derivative of** f **at** c, denoted by $f'(c) = L$, if

$$\forall \varepsilon > 0, \exists \delta_\varepsilon > 0 \text{ s.t. } x \in I \text{ and } 0 < |x-c| < \delta_\varepsilon \implies \left| \frac{f(x) - f(c)}{x - c} - L \right| < \varepsilon.$$

In that case, we say that f is **differentiable** at c. This definition simply states that $f'(c)$ exists if $\lim_{x \to c} \frac{f(x)-f(c)}{x-c}$ exists, and that, in that case,

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}.$$

While $f'(c) \in \mathbb{R}$ (if it exists), $f' : I \to \mathbb{R}$ is the **derivative function**.

Example:

Theorem 39. *If $f : I \to \mathbb{R}$ has a derivative at c, then f is continuous at c.*

Proof.

$\blacksquare$

However, the converse of Theorem 38 does not always hold. For instance, the function $|\cdot| : \mathbb{R} \to \mathbb{R}$ is continuous at $x = 0$, but it has no derivative there as $|x|/x$ has no limit when $x \to 0$.

Continuity is a necessary condition for differentiability, but it is not sufficient.

In fact, it is not too difficult to find functions that are continuous **everywhere**, but **nowhere** differentiable.

Example:

That it took so long to find an example is mostly due to the fact that the definition of a function has evolved a fair amount over the last 200 years.

Theorem 40. *Let I be an interval, $c \in I$, $\alpha \in \mathbb{R}$, and $f, g : I \to \mathbb{R}$ be differentiable at c, with $g(c) \neq 0$. Then*

(a) αf is differentiable at c and $(\alpha f)'(c) = \alpha f'(c)$;

(b) $f + g$ is differentiable at c and $(f + g)'(c) = f'(c) + g'(c)$;

(c) fg is differentiable at c and $(fg)'(c) = f'(c)g(c) + f(c)g'(c)$;

(d) f/g is differentiable at c and $(f/g)'(c) = \dfrac{f'(c)g(c) - f(c)g'(c)}{[g(c)]^2}$.

Proof. In all instances, we compute the limit of the differential quotient, taking into account the fact that f and g are differentiable at c.

(a)

(b)

(c)

(d)

$\blacksquare$

By mathematical induction, we can easily show that

$$\Big[\sum_{i=1}^n f_i\Big]'(c)=\sum_{i=1}^n f_i'(c) \quad \text{and} \quad \Big[\prod_{i=1}^n f_i\Big]'(c)=\sum_{i=1}^n \Big(\prod_{j\neq i} f_j(c)\Big)f_i'(c),$$

if $f_1,\ldots,f_n$ are all differentiable at c. In particular, if $f_1=\cdots=f_n$, then

$$(f^n)'(c)=nf^{n-1}(c)\cdot f'(c).$$

Consider the identity function f. Then for $c\in\mathbb{R}$,

$$f'(c)=\lim_{x\to c}\frac{x-c}{x-c}=1, \implies (f^n)'(x)=nf^{n-1}(x)\cdot f'(x)=nx^{n-1}$$

for all $x\in\mathbb{R}, n\in\mathbb{N}$, which can be extended to $n\in\mathbb{Z}$ using Theorem 40d.

Theorem 41. (Carathéodory)
Let $I = [a, b]$ and $f : I \to \mathbb{R}$. Then f is differentiable at $c \in I$ if and only if $\exists \varphi_c : I \to \mathbb{R}$, continuous at c such that $f(x) - f(c) = \varphi_c(x)(x - c)$, for all $x \in I$. In that case, $\varphi_c(c) = f'(c)$.

Proof.

■

It is important to recognize that φ_c is not, as a function, the same as f', in general – **it is only at c that they can be guaranteed to coincide**, although in certain cases (such as when f is a linear function), $f'(x) = \varphi_c(x)$ for all c in I.

Carethéodory's Theorem can be used to prove the Chain Rule of calculus.

Theorem 42. (Chain Rule)
Let I, J be closed bounded intervals, $g : I \to \mathbb{R}$ and $f : I \to \mathbb{R}$ be functions such that $f(J) \subseteq I$, and let $c \in J$, with $d = f(c)$. If f is differentiable at c and g is differentiable at d, then the composition $g \circ f : J \to \mathbb{R}$ is differentiable at c and $(g \circ f)'(c) = g'(f(c))f'(c) = g'(d)f'(c)$.

Proof.

■

Example:

Example:

5.1.1 – Mean Value Theorem

Let I be an interval. A function $f : I \to \mathbb{R}$ has a **relative maximum** at $c \in I$ if $\exists \delta > 0$ such that

$$f(x) \leq f(c), \quad \forall x \in V_\delta(c) = (c - \delta, c + \delta);$$

it has a **relative minimum** at $c \in I$ if $\exists \delta > 0$ such that

$$f(x) \geq f(c), \quad \forall x \in V_\delta(c) = (c - \delta, c + \delta).$$

If f has either a relative maximum or a relative minimum at c, we say that it has a **relative extremum** at c.

Note that the definition of relative extremum makes no mention of continuity or differentiability.

Theorem 43. *Let $f : [a, b] \to \mathbb{R}$, $c \in (a, b)$. If f has a relative extremum at c and if f is differentiable at c, then $f'(c) = 0$.*

Proof.

■

This result justifies the common practice o looking for relative extrema at roots of the derivative. Note that c is not an endpoint of the interval, and so we must also included them in the search for extrema. What happens if f is not differentiable at c in Theorem 43?

The next theorem has far-reaching consequences.

Theorem 44. (Rolle)
Let $f : [a, b] \to \mathbb{R}$ *be continuous on* $[a, b]$ *and differentiable on* (a, b). *If* $f(a) = f(b) = 0$, $\exists c \in (a, b)$ *such that* $f'(c) = 0$.

Proof.

$\blacksquare$

The Mean Value Theorem is an easy corollary of Rolle's Theorem.

Theorem 45. (Mean Value Theorem)
Let $f : [a, b] \to \mathbb{R}$ *be continuous on* $[a, b]$. *If* f *is differentiable on* (a, b), $\exists c \in (a, b)$ *such that* $f(b) - f(a) = f'(c)(b - a)$.

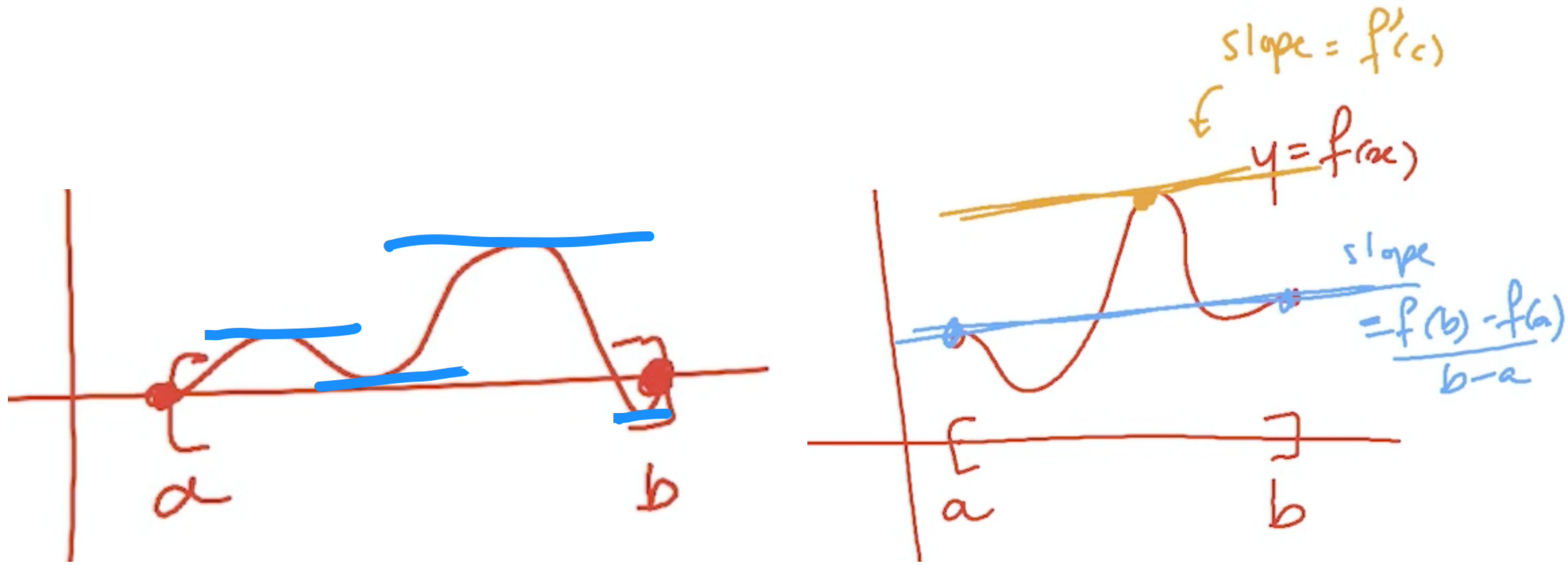
Proof.

$\blacksquare$

Theorem 46. *Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). If $f' \equiv 0$ on (a, b), then f is constant on $[a, b]$.*

Proof.

$\blacksquare$



An illustration of Rolle's Theorem (left); Mean Value Theorem (right).

5.1.2 – Taylor Theorem

Taylor's Theorem is used extensively in applications. It is, in a way, an extension of the Mean Value Theorem to higher order derivatives.

We can naturally obtain the **higher-order derivatives** of a function f by formally applying the differentiation rules repeatedly.

Hence, $f^{(2)} = f'' = (f')'$, $f^{(3)} = f''' = (f'')' = ((f')')'$, etc.

Suppose $f = f^{(0)}$ can be differentiated n times at $x = x_0$. The n**th Taylor polynomial of** f **at** $x = x_0$ is

$$P_n(x; f, x_0) = \sum_{i=0}^{n} \frac{f^{(i)}(x_0)}{i!}(x - x_0)^i.$$

Theorem 47. (Taylor)
Let $n \in \mathbb{N}$ and $f : [a.b] \to \mathbb{R}$ be such that f and its derivatives $f', f'', \ldots, f^{(n)}$ are continuous on $[a, b]$, and $f^{(n+1)}$ exists on (a, b). If $x_0 \in [a, b]$, then for all $x \neq x_0 \in [a, b]$, $\exists c$ between x and x_0 such that

$$f(x) = P_n(x; f, x_0) + \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}.$$

Proof.

■

Example: Use Taylor's Theorem with $n = 2$ to approximate $\sqrt[4]{1+x}$ near $x_0 = 0$ (for $x > -1$).

Solution.

5.1.3 – Relative Extrema

We end this section by giving a characterization of relative extrema using the derivative.

A function $f : I \to \mathbb{R}$ is **increasing** (resp. **decreasing**) if

$$f(x_1) \leq f(x_2), \quad \text{(resp. } f(x_1) \geq f(x_2)) \quad \forall x_1 \leq x_2 \in I.$$

If the inequalities are strict, then the function is **strictly increasing** (resp. **strictly decreasing**).

A function that is either increasing or decreasing (exclusively) is **monotone**.

If the function is also differentiable, then a link exists.

Theorem 48. *Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$, differentiable on (a, b). Then f is increasing on $[a, b]$ if and only if $f' \geq 0$ on (a, b).*

Proof.

$\blacksquare$

Theorem 48 holds for decreasing functions as well (after having made the obvious changes to the statement). If we switch to strictly monotone functions, only one direction holds in all cases – which one?

The next result is a celebrated result from calculus.

Theorem 49. (First Derivative Test)
Let f be continuous on $[a,b]$ and let $c \in (a,b)$. Suppose f is differentiable on (a,c) and on (b,c), but not necessarily at c. Then

(a) if $\exists V_\delta(c) \subseteq [a,b]$ such that $f'(x) \geq 0$ for $c-\delta < x < c$ and $f'(x) \leq 0$ for $c < x < c+\delta$, then f has a relative maximum at c;

(b) if $\exists V_\delta(c) \subseteq [a,b]$ such that $f'(x) \leq 0$ for $c-\delta < x < c$ and $f'(x) \geq 0$ for $c < x < c+\delta$, then f has a relative minimum at c.

Proof.

■

The converse of the First Derivative Test is not necessarily true. For instance, the function defined by

$$f(x) =$$

has an absolute minimum at $x = 0$, but it has derivatives of either sign on either side of any neighbourhood of $x = 0$.

We end this section with a rather surprising result.

Theorem 50. (DARBOUX)
Let $f : [a, b] \to \mathbb{R}$ be differentiable, continuous on $[a, b]$ and let k be strictly confined between $f'(a)$ and $f'(b)$. Then $\exists c \in (a, b)$ with $f'(c) = k$.

Proof.

■

Darboux's Theorem states that the derivative of a continuous function, which needs not be continuous, nevertheless satisfies the intermediate value property.

There are a number of other results which could be shown about differentiable functions, but that are left as exercises. Let $f : (a,b) \to \mathbb{R}$ be differentiable on (a,b), with $f'(x) \neq 0$. Then

- f is monotone on (a,b) and $f((a,b))$ is an open interval (α,β);
- f has an **inverse** $f^{-1} : (\alpha,\beta) \to \mathbb{R}$ such that
$$f^{-1}(f(x)) = x, \quad f(f^{-1}(y)) = y, \quad \forall x \in (a,b), y \in (\alpha,\beta),$$
- f^{-1} is differentiable on (α,β), with
$$(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}, \quad \forall y \in (\alpha,\beta).$$

5.2 – Riemann Integral

Calculus as a discipline only took flight after Newton announced his **theory of fluxions**. With Leibniz' independent discovery that the reversal of the process for fining tangents lea to areas under curves, integration was born.

Riemann was the first to study integration as a process **separate** from differentiation.

We start by studying the integration of a functions $\mathbb{R} \to \mathbb{R}$. Later on, you will tackle integration of functions $\mathbb{R}^n \to \mathbb{R}$; and, eventually, of functions $\mathbb{R}^n \to \mathbb{R}^n$.

Let $I = [a, b]$. A **partition** P of $[a, b]$ is a subset $P = \{x_0, \ldots, x_n\} \subseteq I$ such that

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b.$$

If $f : I \to \mathbb{R}$ is bounded and P is a partition of I, the sum

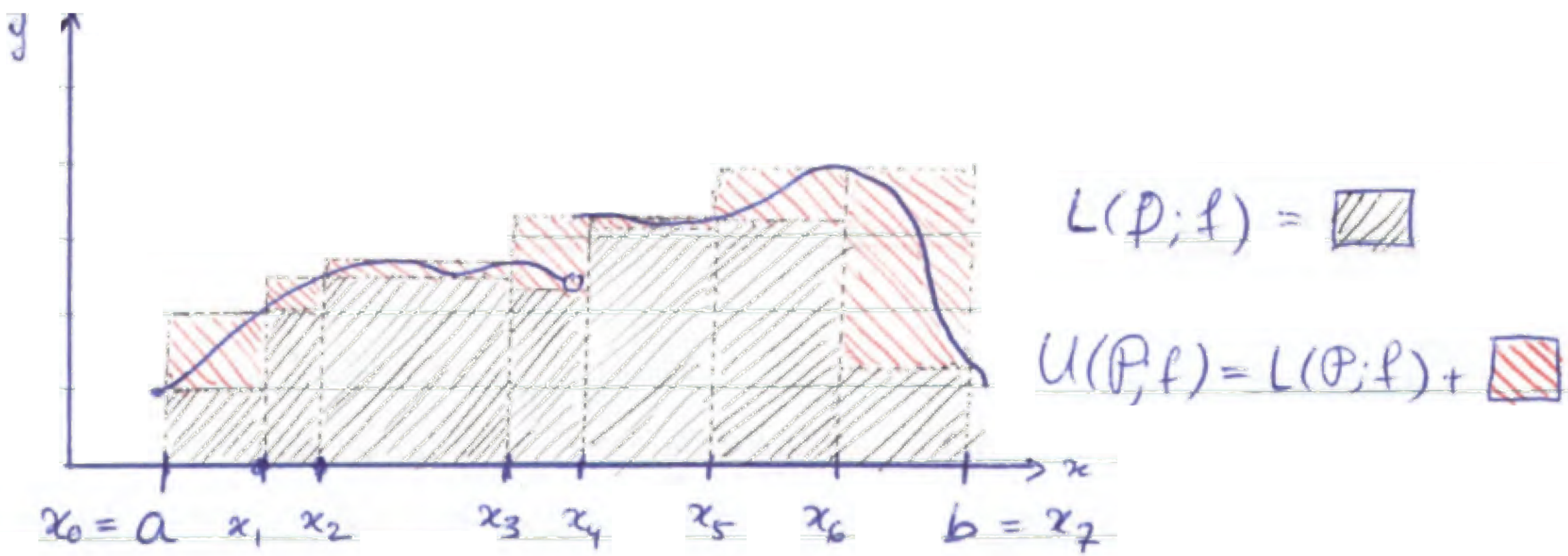
$$L(P; f) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}) < \infty, \quad U(P; f) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}) < \infty,$$

where

$$m_i = \inf\{f(x) \mid x \in [x_{i-1}, x_i]\}, \quad M_i = \sup\{f(x) \mid x \in [x_{i-1}, x_i]\}, \quad 1 \leq i \leq n$$

are the **lower** and the **upper sum of** f **corresponding to** P, respectively.

If $f : I \to \mathbb{R}_0^+$, we can give a graphical representation of these sums.



$L(P; f)$ is the area of the union of the rectangles with base $[x_{k-1}, x_k]$ and height m_k, and $U(P; f)$ is the area of the union of the rectangles with base $[x_{k-1}, x_k]$ and height M_k.

In general, a partition Q of I is a **refinement** of a partition P of I if $P \subseteq Q$.

Example:

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Lemma 1. *Let $I = [a, b]$ and $f : I \to \mathbb{R}$ be bounded. Then*

(a) $L(P; f) \leq U(P; f)$ for any partition P of I;

(a) $L(P; f) \leq L(Q; f)$ and $U(Q; f) \leq L(Q; f)$ for any refinement $Q \supseteq P$ of I, and

(a) $L(P_1; f) \leq U(P_2; f)$ for any pair of partitions P_1, P_2 of I.

Proof.

(a)

(b)

(c)

$\blacksquare$

Let $I = [a, b]$ and $f : I \to \mathbb{R}$ be bounded. The **lower integral of f on I** is the number

$$L(f) = \sup\{L(P; f) \mid P \text{ a partition of } I\}.$$

The **upper integral of f on I** is the number

$$U(f) = \inf\{U(P; f) \mid P \text{ a partition of } I\}.$$

Since f is bounded on I, $\exists m, M$ such that $m \leq f(x) \leq M$ for all $x \in I$. Consider the **trivial partition** $P_0 = \{a, b\}$. Since any partition P of I is a refinement of P_0, we thus have

$$L(P; f) \leq U(P_0; f) \leq M(b - a) \quad \text{and} \quad U(P; f) \geq L(P_0; f) \geq m(b - a).$$

Thus $L(f)$, $U(f)$ exist, by completeness. But we can say more.

Theorem 51. *Let $f : [a, b] \to \mathbb{R}$ be bounded. Then $L(f) \leq U(f)$.*

Proof.

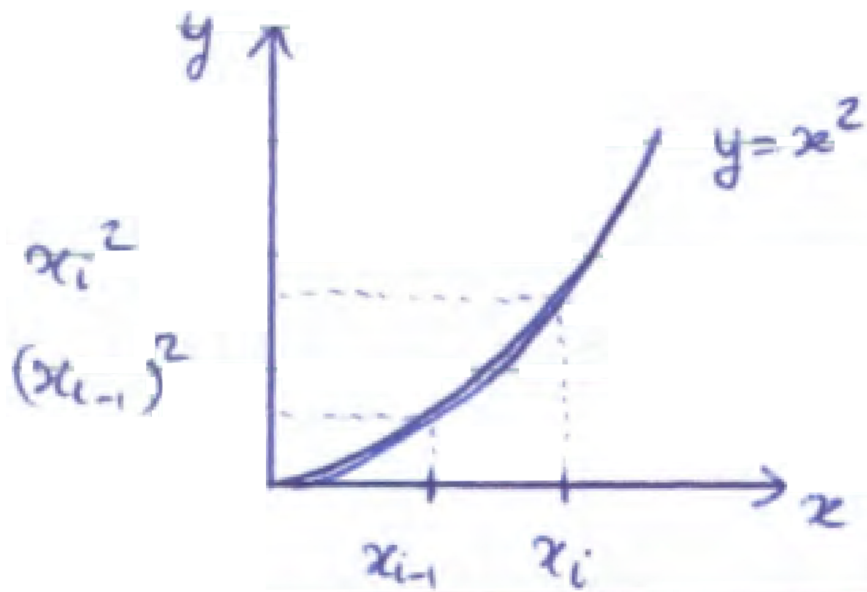
■

When $L(f) = U(f)$, we say that f is **Riemann-integrable** on $[a,b]$; the **integral of** f **on** $[a,b]$ is the real number

$$L(f) = U(f) = \int_a^b f = \int_a^b f(x)\,dx.$$

By convention, we define $\int_a^b f = -\int_b^a f$ when $b < a$. Note that $\int_a^a f = 0$ for all bounded functions f.

Example: Show directly that the function defined by $h(x) = x^2$ is Riemann-integrable on $[a, b]$, $b > a \geq 0$. Furthermore show that $\int_a^b h = \frac{b^3 - a^3}{3}$.



Proof.

$\blacksquare$

Example: Show directly that the **Dirchlet function** defined by

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$$

is not Riemann-integrable on $[0, 1]$.

Proof.

$\blacksquare$

This last example underlines some of the shortcomings of the Riemann integral.

The integral of this function should really be 0 on $[0, 1]$: the set $\mathbb{R} \setminus \mathbb{Q}$ is so much larger than $\mathbb{Q}$ that whatever happens on $\mathbb{Q}$ should largely be irrelevant.

There are various theories of integration – as we shall see in a later chapter, the Lebesgue-Borel integral of f on $[0, 1]$ is indeed 0.

Other issues arise with the Riemann integral, which we will discuss in the coming sections.

5.2.1 – Riemann Criterion

For the time being, we will focus on two **fundamental questions** associated with the Riemann integral of a function over an interval $[a, b]$:

But the direct approach is cumbersome, even for the simplest of functions.

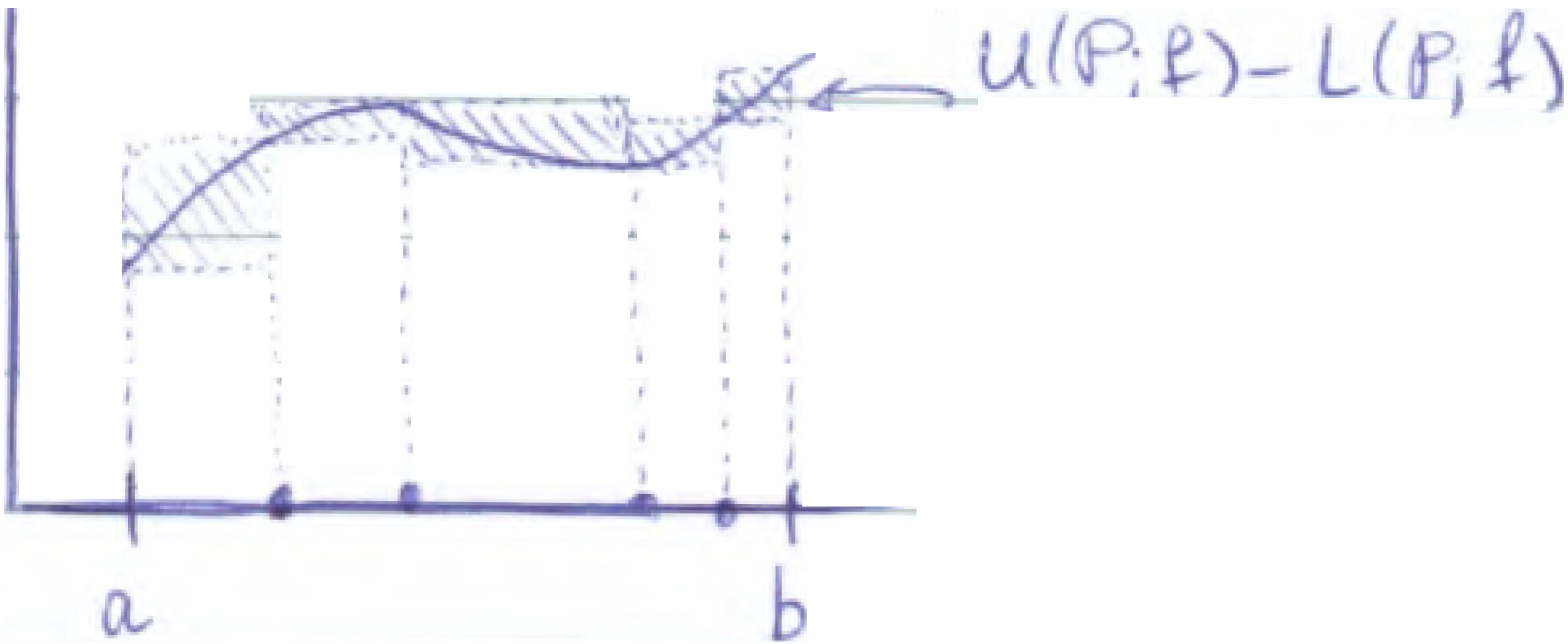
The following result allows us to bypass the need to compute $L(f)$ and $U(f)$ to determine if a function is Riemann-integrable or not.

Theorem 52. (RIEMANN'S CRITERION)
Let $I = [a, b]$ and $f : I \to \mathbb{R}$ be a bounded function. Then f is Riemann-integrable if and only if $\forall \varepsilon > 0$, $\exists P_\varepsilon$ a partition of I such that the lower sum and the upper sum of f corresponding to P_ε satisfy $U(P_\varepsilon; f) - L(P_\varepsilon; f) < \varepsilon$.

Proof.

$\blacksquare$

The Riemann Criterion is illustrated below for a continuous function:



The smaller the shaded area is, the closer $U(P;f)$ and $L(P;f)$ are to $\int_a^b f$.

There are 2 instances where the Riemann-integrability of a function f on $[a, b]$ is guaranteed: when f is **monotone**, and when it is **continuous**.

Theorem 53. *Let $I = [a, b]$ and $f : I \to \mathbb{R}$ be a monotone function on I. Then f is Riemann-integrable on I.*

Proof.

$=$

■

Theorem 54. *Let $I = [a, b]$ and $f : I \to \mathbb{R}$ be continuous, with $a < b$. Then f is Riemann-integrable on I.*

Proof.

$\blacksquare$

5.2.2 – Properties of the Riemann Integral

The Riemann integral has a whole slew of interesting properties.

Theorem 55. (Properties of the Riemann Integral)
Let $I = [a, b]$ and $f, g : I \to \mathbb{R}$ be Riemann-integrable on I. Then

(a) $f + g$ is Riemann-integrable on I, with $\int_a^b (f + g) = \int_a^b f + \int_a^b g$;

(b) if $k \in \mathbb{R}$, $k \cdot f$ is Riemann-integrable on I, with $\int_a^b k \cdot f = k \int_a^b f$;

(c) if $f(x) \leq g(x)\ \forall x \in I$, then $\int_a^b f \leq \int_a^b g$, and

(d) if $|f(x)| \leq K\ \forall x \in I$, then $\left|\int_a^b f\right| \leq K(b - a)$.

Proof. We use a variety of pre-existing results.

(a)

(b)

(c)

(d)

■

When all the functions involved are non-negative, these results and the next one are compatible with the area under the curve from calculus (more on this in the next section).

Theorem 56. (Additivity of the Riemann Integral)
Let $I = [a, b]$, $c \in (a, b)$, and $f : I \to \mathbb{R}$ be bounded on I. Then f is Riemann-integrable on I if and only if it is Riemann-integrable on $I_1 = [a, c]$ and on $I_2 = [c, b]$. When that is the case, $\int_a^b f = \int_a^c f + \int_c^b f$.

Proof.

$\blacksquare$

Theorem 57. (Composition Theorem for Integrals)
Let $I = [a, b]$ *and* $J = [\alpha, \beta]$*,* $f : I \to \mathbb{R}$ *Riemann-integrable on* I*,* $\varphi : J \to \mathbb{R}$ *continuous on* J *and* $f(I) \subseteq J$*. Then* $\varphi \circ f : I \to \mathbb{R}$ *is Riemann-integrable on* I*.*

Proof.

$\blacksquare$

The proof of Composition Theorem requires the intervals I and J to be closed, as the following example shows.

Example:

Note, however, that there are examples of functions defined on open intervals for which the conclusion of the Composition Theorem holds.

Example:

The Composition Theorem can be used to show a variety of results.

Theorem 58. *Let* $I = [a, b]$ *and* $f, g : I \to \mathbb{R}$ *be Riemann-integrable on* I*. Then* fg *and* $|f|$ *are Riemann-integrable on* I*, and* $\left|\int_a^b f\right| \leq \int_a^b |f|$*.*

Proof.

■

Note that even though the product of Riemann-integrable functions is itself Riemann-integrable there is no simple way to express $\int_a^b fg$ in terms of $\int_a^b f$ and $\int_a^b g$.

A surprising result is that the composition of Riemann-integrable functions need not be Riemann-integrable. A counter example is provided below.

Example:

5.2.3 – Fundamental Theorem of Calculus

With Descartes' creation of analytical geometry, it became possible to find the **tangents** to curves that could be described **algebraically**.

Fermat then showed the connection between that problem and the problem of finding the **maximum/minimum** of a (continuous) function.

Newton and Leibniz, in the 1680s, then discovered that computing the **area underneath a curve** is exactly the opposite of finding the tangent.

Calculus provided a general framework to solve problems that had hiterto been very difficult to solve (and even then, only in specific circumstance).

In this section, we study the connection between these concepts.

Theorem 59. (Fundamental Theorem of Calculus, 1st version) *Let $I = [a, b]$, $f : I \to \mathbb{R}$ be Riemann-integrable on I, and $F : I \to \mathbb{R}$ be such that F is continuous on I and differentiable on (a, b). If $F'(x) = f(x)$ for all $x \in (a, b)$, then $\int_a^b f = F(b) - F(a)$.*

Proof.

$\blacksquare$

This classical calculus result is quite useful in applications.

We will see in a later chapter that the Fundamental Theorem of Calculus (1st version) is a special case of the general result known as Stokes' Theorem.

Theorem 60. (FUNDAMENTAL THEOREM OF CALCULUS, 2ND VERSION) *Let $I = [a, b]$, $f : I \to \mathbb{R}$ be Riemann-integrable on I, and define a function $F : I \to \mathbb{R}$ by $F(x) = \int_a^x f$. Then F is continuous on I. Furthermore, if f is continuous at $c \in (a, b)$, then F is differentiable at c and $F'(c) = f(c)$.*

Proof.

The first version of the Fundamental Theorem Calculus provides a justification of the **method used to evaluate definite integrals** in calculus; the second version, which allows the upper bound of the Riemann integral to vary, provides a basis for finding antiderivatives.

Let $I = [a, b]$ an $f : I \to \mathbb{R}$. An **antiderivative** of f on I is a differentiable function $F : I \to \mathbb{R}$ such that $F'(x) = f(x)$ for all $x \in I$.

If f is Riemann-integrable on I, the function $F : I \to \mathbb{R}$ defined by $F(x) = \int_a^x f$ for $x \in I$ is the **indefinite integral of f on I.**

If f is Riemann-integrable on I and if F is an antiderivative of f on I, then

$$\int_a^b f = F(b) - F(a).$$

However, Riemann-integrable functions on I may not have antiderivatives on I (such as the signum and Thomae's functions), and functions with antiderivatives may not be Riemann-integrable on I (such as the reciprocal of the square root function on $[0, 1]$).

If f is Riemann-integrable on I, then $F(x) = \int_a^x f$ exists. Moreover, if f is continuous on I, than F is an antiderivative of f on I, since $F'(x) = f(x)$ for all $x \in I$.

Continuous functions thus always have antiderivatives (even if they can't be expressed using elementary functions).

But if f is not continuous on I, the indefinite integral F may not be an antiderivative of f on I – it may fail to be differentiable at certain points of I, or F' may exists but be different from f at various points of I.

5.2.4 – Evaluation of Integrals

We complete this chapter by presenting some common methods used to evaluate integrals.

We provide the proof of two well-known calculus results.

Theorem 61. (Integration by Parts)
Let $f,g:[a,b]\to\mathbb{R}$ both be Riemann-integrable on $[a,b]$, with antiderivatives $F,G:[a,b]\to\mathbb{R}$, respectively. Then

$$\int_a^b Fg = F(b)G(b)-F(a)G(a)-\int_a^b fG.$$

Proof.

■

Theorem 62. (First Substitution Theorem)
Let $J = [\alpha, \beta]$, and $\varphi \to \mathbb{R}$ be a function with a continuous derivative on J. If $f : I \to \mathbb{R}$ is continuous on $I = [a, b] \supseteq \varphi(J)$, then

$$\int_{\alpha}^{\beta} (f \circ \varphi)\varphi' = \int_{\varphi(\alpha)}^{\varphi(\beta)} f.$$

Proof.

■

Theorem 63. (SECOND SUBSTITUTION THEOREM)
Let $J = [\alpha, \beta]$, and $\varphi \to \mathbb{R}$ be a function with a continuous derivative on J and such that $\varphi' \neq 0$ on J. Let $I = [a, b] \supseteq \varphi(J)$, and $\psi : I \to \mathbb{R}$ be the inverse of φ (which exists as φ is montoone). If $f : I \to \mathbb{R}$ is continuous on I, then

$$\int_\alpha^\beta f \circ \varphi = \int_{\varphi(\alpha)}^{\varphi(\beta)} f \psi'.$$

Theorem 64. (Mean Value Theorem for Integrals)
Let $I = [a,b]$, $f : I \to \mathbb{R}$ be continuous on I, and $p : I \to \mathbb{R}$ be Riemann-integrable on I, with $p \geq 0$ on I. Then $\exists c \in (a,b)$ such that

$$\int_a^b fp = f(c) \int_a^b p.$$

Theorem 65. (Squeeze Theorem for Integrals)
Let $I = [a,b]$ and $f \leq g \leq h : I \to \mathbb{R}$ be bounded on I. If f,h are Riemann-integrable on I with $\int_a^b f = \int_a^b h$, then g is Reimann-integrable on I and $\int_a^b g = \int_a^b f = \int_a^b h$.

The proofs of these three last theorems are left as an exercise.

5.3 – Exercises

1. Use the definition to find the derivative of the function defined by $g(x) = \frac{1}{x}$, $x \in \mathbb{R}$, $x \neq 0$.
2. Prove that the derivative of an even differentiable function is odd, and vice-versa.
3. Let $a > b > 0$ and $n \in \mathbb{N}$ with $n \geq 2$. Show that $a^{1/n} - b^{1/n} < (a-b)^{1/n}$.
4. Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). Show that if $\lim_{x\to a} f'(x) = A$, then $f'(a)$ exists and equals A.
5. If $x > 0$, show $1 + \frac{1}{2}x - \frac{1}{8}x^2 \leq \sqrt{1+x} \leq 1 + \frac{1}{2}x$.
6. Let $a \in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x^2 & \text{if } x \geq 0, \\ ax & \text{if } x < 0. \end{cases}$$

   For which values of a is f differentiable at $x = 0$? For which values of a is f continuous at $x = 0$?

7. Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Show that f is Lipschitz if and only if f' is bounded on (a, b).
8. Prove that $\int_0^1 g = \frac{1}{2}$ if
$$g(x) = \begin{cases} 1 & x \in (\frac{1}{2}, 1] \\ 0 & x \in [0, \frac{1}{2}] \end{cases}.$$
Is that still true if $g(\frac{1}{2}) = 7$ instead?
9. Let $f : [a, b] \to \mathbb{R}$ be bounded and s.t. $f(x) \geq 0 \ \forall x \in [a, b]$. Show $L(f) \geq 0$.
10. Let $f : [a, b] \to \mathbb{R}$ be increasing on $[a, b]$. If P_n partitions $[a, b]$ into n equal parts, show that
$$0 \leq U(P_n; f) - \int_a^b f \leq \frac{f(b) - f(a)}{n}(b - a).$$
11. Let $f : [a, b] \to \mathbb{R}$ be an integrable function and let $\varepsilon > 0$. If P_ε is the partition whose existence is asserted by the Riemann Criterion, show that $U(P; f) - L(P; f) < \varepsilon$ for all refinement P of P_ε.
12. Let $a > 0$ and $J = [-a, a]$. Let $f : J \to \mathbb{R}$ be bounded and let $\mathcal{P}^*$ be the set of symmetric partitions of J that contain 0. Show $L(f) = \sup\{L(P; f) : P \in \mathcal{P}^*\}$.

13. Let $a > 0$ and $J = [-a, a]$. Let f be integrable on J. If f is even (i.e. $f(-x) = f(x)$ for all x), show that

$$\int_{-a}^{a} f = 2\int_{0}^{a} f.$$

If f is odd (i.e. $f(-x) = -f(x)$ for all x), show that $\displaystyle\int_{-a}^{a} f = 0$.

14. Give an example of a function $f : [0, 1] \to \mathbb{R}$ that is not integrable on $[0, 1]$, but s.t. $|f|$ is integrable on $[0, 1]$.
15. Let $f : [a, b] \to \mathbb{R}$ be integrable on $[a, b]$. Show $|f|$ is integrable on $[a, b]$ directly (without using a result seen in class).
16. If f is integrable on $[a, b]$ and $0 \leq m \leq f(x) \leq M$ for all $x \in [a, b]$, show that

$$m \leq \left[\frac{1}{b-a}\int_{a}^{b} f^2\right]^{1/2} \leq M.$$

17. If f is continuous on $[a,b]$ and $f(x) \geq 0$ for all $x \in [a,b]$, show there exists $c \in [a,b]$ s.t.
$$f(c) = \left[\frac{1}{b-a}\int_a^b f^2\right]^{1/2}.$$
18. If f is continuous on $[a,b]$ and $f(x) > 0$ for all $x \in [a,b]$, show that $\frac{1}{f}$ is integrable on $[a,b]$.
19. Let f be continuous on $[a,b]$. Define $H : [a,b] \to \mathbb{R}$ by
$$H(x) = \int_x^b f \quad \text{for all } x \in [a,b].$$
Find $H'(x)$ for all $x \in [a,b]$.
20. Suppose $f : [0,\infty) \to \mathbb{R}$ is continuous and $f(x) \neq 0$ for all $x > 0$. If
$$(f(x))^2 = 2\int_0^x f \quad \text{for all } x > 0,$$
show that $f(x) = x$ for all $x \geq 0$.

21. Let $f, g : [a, b] \to \mathbb{R}$ be continuous and s.t.

$$\int_a^b f = \int_a^b g.$$

Show that there exists $c \in [a, b]$ s.t. $f(c) = g(c)$.

22. If $f : [0, 1] \to \mathbb{R}$ is continuous and $\int_0^x f = \int_x^1 f$ for all $x \in [0, 1]$, show that $f \equiv 0$.

23. Let $f : [0, 3] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x & x \in [0,1) \\ 1 & x \in [1,2) \\ x & x \in [2,3] \end{cases}.$$

Find $F : [0, 3] \to \mathbb{R}$, where

$$F(x) = \int_0^x f.$$

Where is F differentiable? What is F' there?

24. Let $f : [a, b] \to \mathbb{R}$ be continuous, $f \geq 0$ on $[a, b]$, and $\int_a^b f = 0$. Show that $f \equiv 0$ on $[a, b]$.
25. Let $f : [a, b] \to \mathbb{R}$ be continuous and let $\int_a^b f = 0$. Show $\exists c \in [a, b]$ such that $f(c) = 0$.
26. Compute $\displaystyle\frac{d}{dx}\int_{-x}^{x} e^{t^2}\, dt$.
27. Let $f : [a, b] \to \mathbb{R}$ be Riemann-integrable on $[a + \delta, b]$ and unbounded in the interval $(a, a + \delta)$ for every $0 < \delta < b - a$. Define

$$\int_a^b f = \lim_{\delta\to 0^+} \int_{a+\delta}^b f,$$

where $\delta \to 0^+$ means that $\delta \to 0$ and $\delta > 0$. A similar construction allows us to define

$$\int_a^b g = \lim_{\delta\to 0^+} \int_a^{b-\delta} g.$$

Such integrals are said to be **improper**; when the limits exist, they are further said to be **convergent**.

How can the expression

$$\int_0^1 \frac{1}{\sqrt{|x|}}\,dx$$

be interpreted as an improper integral? Is it convergent? If so, what is its value?

28. For which values of s does the integral $\int_0^1 x^s\,dx$ converge?

Solutions

1. **Proof.**

2. **Proof.**

3. **Proof.**

4. **Proof.**

5. **Proof.**

6. **Solution.**

7. **Proof.**

8. **Proof.**

9. **Proof.**

10. **Proof.**

11. **Proof.**

12. **Proof.**

13. **Proof.**

14. **Solution.**

15. **Proof.**

16. **Proof.**

17. **Proof.**

18. **Proof.**

19. **Proof.**

20. **Proof.**

21. **Proof.**

22. **Proof.**

23. **Proof.**

24. **Proof.**

25. **Proof.**

26. **Solution.**

Solution.

27. **Solution.**