MAT 2125
Elementary Real Analysis

Chapter 6
Sequences of Functions

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Overview

We now look at sequences of functions, which arise naturally in analysis and its applications.

In particular, we will

- discuss two types of convergence (pointwise and uniform), and
- prove some limit interchange theorems.

Outline

6.1 – Pointwise and Uniform Convergence

Let $A \subseteq \mathbb{R}$ and $(f_n)_n$ be a **sequence of functions** $f_n : A \to \mathbb{R}$.

The sequence $(f_n(x))_n$ may converge for some $x \in A$ and diverge for others.

Let $A_0 = \{x \in A \mid (f_n(x))_n \text{ converges}\} \subseteq A$. For each $x \in A_0$, $(f_n(x))$ converges to a unique limit

$$f(x) = \lim_{n\to\infty} f(x),$$

the **pointwise limit** of (f_n), which we denote by $f_n \to f$ on A_0.

Examples:

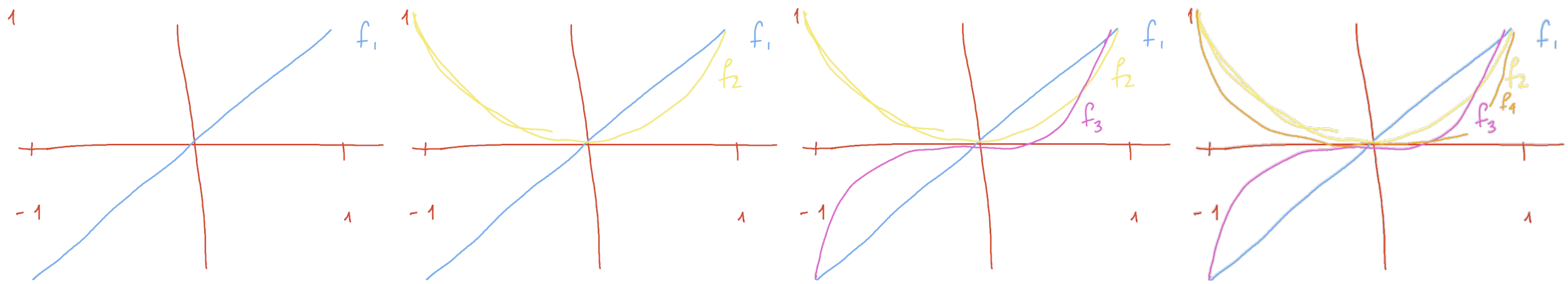
1. Let $f_n : \mathbb{R} \to \mathbb{R}$ be defined by $f_n(x) = \frac{x}{n}$ for all $n \in \mathbb{N}, x \in \mathbb{R}$, and let f be the zero function on $\mathbb{R}$. Show that $f_n \to f$ on $\mathbb{R}$.

   Proof.

   ■

2. Let $f_n : \mathbb{R} \to \mathbb{R}$ be defined by $f_n(x) = x^n$ for all $n \in \mathbb{N}, x \in \mathbb{R}$, and let f be the zero function on $\mathbb{R}$, except at $x = 1$ where $f(1) = 1$. Show that $f_n \to f$ on $(-1, 1]$.

Proof.



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3. Let $f_n : \mathbb{R} \to \mathbb{R}$ be defined by $f_n(x) = \frac{x^2+nx}{n}$ for all $n \in \mathbb{N}, x \in \mathbb{R}$, and let f be the identity function on $\mathbb{R}$. Show that $f_n \to f$ on $\mathbb{R}$.

   Proof. $\blacksquare$

A sequence of functions $(f_n : A \to \mathbb{R})$ **converges uniformly on** $A_0 \subseteq A$ **to** $f : A_0 \to \mathbb{R}$, denoted by $f_n \rightrightarrows f$ on A_0, if the threshold $N_{\varepsilon,x} \in \mathbb{N}$ in the pointwise definition is in fact **independent** of $x \in A_0$:

$$\forall \varepsilon > 0,\ \exists N_\varepsilon \in \mathbb{N} \text{ such that } n > N_\varepsilon \text{ and } x \in A_0 \implies |f_n(x) - f(x)| < \varepsilon.$$

The distinction between pointwise and uniform convergence is not unlike that between continuity and uniform continuity: convergence is uniform if the threshold is the same for all $x \in A_0$.

Clearly, if $f_n \rightrightarrows f$ on A_0, then $f_n \to f$ on A_0, but the converse is not necessarily true.

Examples:

1. Show that the sequence $f_n : [1,2] \to \mathbb{R}$ defined by $f_n(x) = \frac{\sin x}{nx}$ for $n \in \mathbb{N}$ converges uniformly to the zero function on $[1,2]$.

   Proof.

   ■

2. Let $f_n : \mathbb{R} \to \mathbb{R}$ be defined by $f_n(x) = x^n$ for all $n \in \mathbb{N}, x \in \mathbb{R}$, and let f be the zero function on $\mathbb{R}$, except at $x = 1$ where $f(1) = 1$. Show that $f_n \not\rightrightarrows f$ on $(-1, 1]$.

   Proof.

   ■

A sequence of functions f_n does not converge uniformly to f on A_0 if

$\exists \varepsilon_0 > 0$ with $(f_{n_k}) \subseteq (f_n)$ and $(x_k) \subseteq A_0$ s.t. $|f_{n_k}(x_k) - f(x_k)| \geq \varepsilon_0$, $\forall k \in \mathbb{N}$.

The definition of uniform convergence is only ever useful if a candidate for a uniform limit is available, a situation that we have encountered before.

Theorem 66. (Cauchy's Criterion for Sequences of Functions) *Let* $f_n : A \to \mathbb{R}$*, for all* $n \in \mathbb{N}$*. Then,* $f_n \rightrightarrows f$ *on* $A_0 \subseteq A$ *if and only if* $\forall \varepsilon > 0$*,* $\exists N_\varepsilon \in \mathbb{N}$ *(indep. of* $x \in A_0$*) such that* $|f_m(x) - f_n(x)| < \varepsilon$ *whenever* $m \geq n > N_\varepsilon \in \mathbb{N}$ *and* $x \in A_0$*.*

Proof.

$\blacksquare$

Example: Let $f_n : [0, 1] \to \mathbb{R}$ be the sequence of functions defined by

$$f_n(x) = \begin{cases} nx, & x \in [0, 1/n] \\ 2 - nx, & x \in [1/n, 2/n] \\ 0 & x \in [2/n, 1] \end{cases}$$

for all $n \in \mathbb{N}$. Let $f : [0, 1] \to \mathbb{R}$ be the zero function on $[0, 1]$. Show that $f_n \to f$ on $[0, 1]$ but $f_n \not\rightrightarrows f$ on $[0, 1]$.

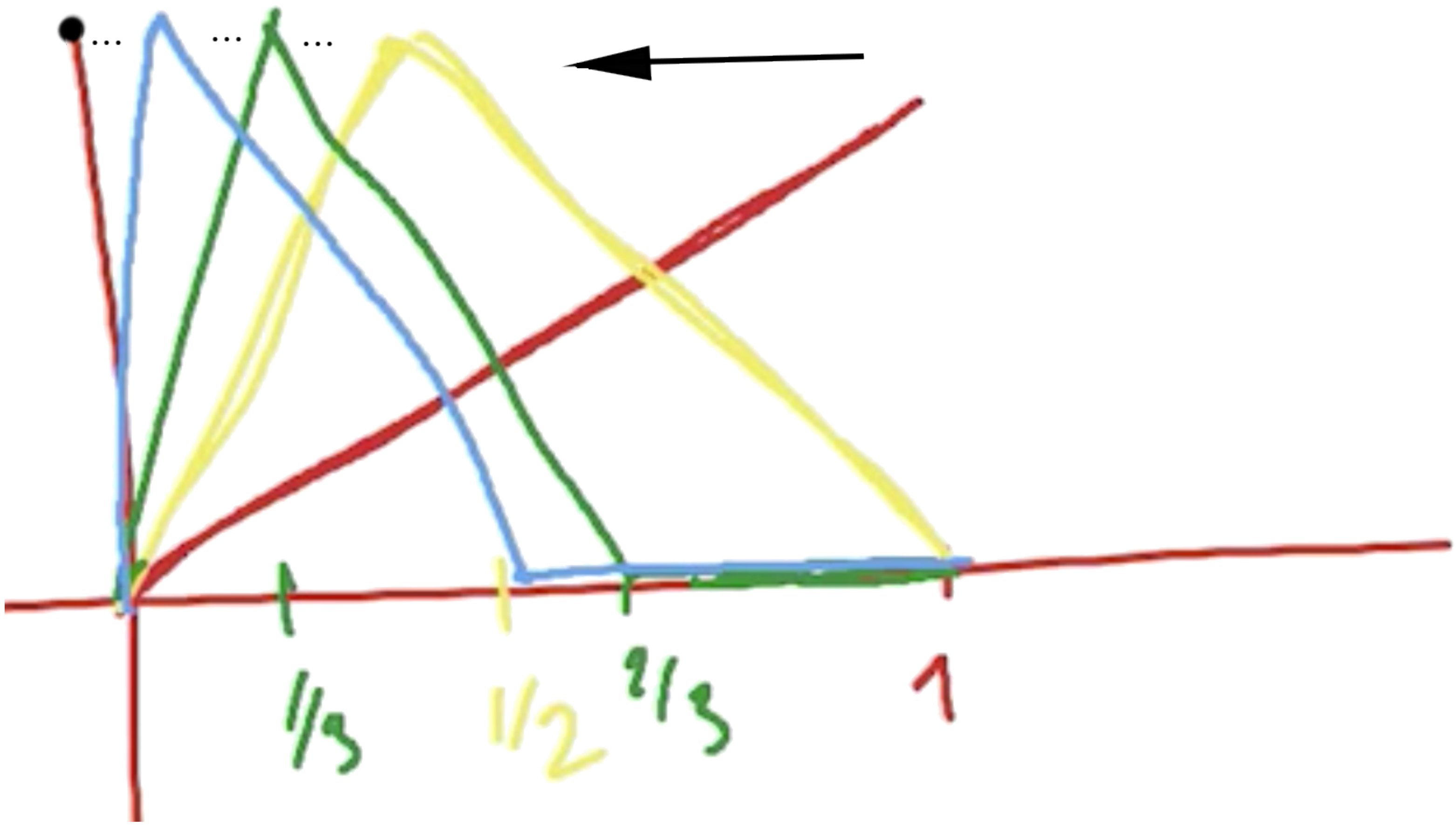
Proof.

■

The fact that we have to separate the proof for pointwise convergence into distinct argument depending on the value of x is a strong indication that the convergence cannot be uniform (although it could be that it was possible to do a one-pass proof and that the insight escaped us...)

Intuitively, we can think of the convergence process in the last example as being a flattening process: what happens to the tents' peak as $n \to \infty$?

The fact that we have to "break" the tents in order to get to the pointwise limit is another indication that the convergence cannot be uniform.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6.2 – Limit Interchange Theorems

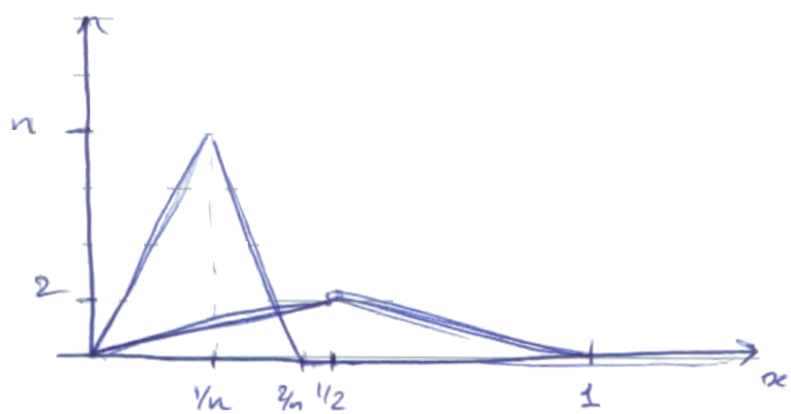
It is often necessary to know if the limit f of a sequence of functions (f_n) is continuous, differentiable, or Riemann-integrable. It is not always the case, even when the f_n are continuous, differentiable, or Riemann-integrable.

Examples:

1.

2.

3.



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Note that none of the "convergences" in the previous example are uniform on $[0,1]$. When the convergence $f_n \rightrightarrows f$ on A is uniform, then if the f_n are

- continuous on A, so is f;
- differentiable on A, so is f, with

$$f' = \frac{d}{dx}\Big[\lim_{n\to\infty} f_n\Big] = \lim_{n\to\infty}\left[\frac{d}{dx} f_n\right] = \lim_{n\to\infty} f_n';$$

- Riemann-integrable on A, then so is f, with

$$\int_A f = \int_A \lim_{n\to\infty} f_n = \lim_{n\to\infty} \int_A f_n.$$

We finish this chapter by proving three **Limit Interchange Theorems**.

Theorem 67. *Let $f_n : A \to \mathbb{R}$ be continuous on A for all $n \in \mathbb{N}$. If $f_n \rightrightarrows f$ on A, then f is continuous on A.*

Proof.

$\blacksquare$

Theorem 68. *Let $f_n : [a, b] \to \mathbb{R}$ be a sequence of differentiable functions on $[a, b]$ such that $\exists x_0 \in [a, b]$ with $f_n(x_0) \to z_0$, and $f_n'' \rightrightarrows g$ on $[a, b]$. Then $f_n \rightrightarrows f$ on $[a, b]$ for some function $f : [a, b] \to \mathbb{R}$ such that $f' = g$.*

Proof.

■

Theorem 69. *Let* $f_n : [a,b] \to \mathbb{R}$ *be Riemann-integrable on* $[a,b]$ *for all* $n \in \mathbb{N}$*. If* $f_n \rightrightarrows f$ *on* $[a,b]$*, then* f *is Riemann-integrable on* $[a,b]$ *and*

$$\int_a^b f = \lim_{n\to\infty} \int_a^b f_n.$$

Proof.

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6.3 – Exercises

1. Show that $\lim_{n\to\infty} \dfrac{nx}{1+n^2x^2} = 0$ for all $x \in \mathbb{R}$.
2. Show that if $f_n(x) = x + \frac{1}{n}$ and $f(x) = x$ for all $x \in \mathbb{R}$, $n \in \mathbb{N}$, then $f_n \rightrightarrows f$ on $\mathbb{R}$ but $f_n^2 \not\rightrightarrows g$ on $\mathbb{R}$ for any function g.
3. Let $f_n(x) = \frac{1}{(1+x)^n}$ for $x \in [0,1]$. Denote by f the pointwise limit of f_n on $[0,1]$. Does $f_n \rightrightarrows f$ on $[0,1]$?
4. Let (f_n) be the sequence of functions defined by $f_n(x) = \frac{x^n}{n}$, for $x \in [0,1]$ and $n \in \mathbb{N}$. Show that (f_n) converges uniformly to a differentiable function $f : [0,1] \to \mathbb{R}$, and that the sequence (f_n') converges pointwise to a function $g : [0,1] \to \mathbb{R}$, but that $g(1) \neq f'(1)$.
5. Show that $\lim_{n\to\infty} \displaystyle\int_1^2 e^{-nx^2}\,dx = 0$.

6. Show that $\lim_{n\to\infty} \int_{\pi/2}^{\pi} \frac{\sin(nx)}{nx}\, dx = 0$.
7. Show that if $f_n \rightrightarrows f$ on $[a, b]$, and each f_n is continuous, then the sequence of functions $(F_n)_n$ defined by

$$F_n(x) = \int_a^x f_n(t)\, dt$$

also converges uniformly on $[a, b]$.

Solutions

1. **Proof.**

2. **Proof.**

3. **Proof.**

4. **Proof.**

5. **Proof.**

6. **Proof.**

7. **Proof.**