

Mathematical Analysis

Chapter 7 Series of Functions

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Overview

We discuss a specific type of sequence: the series.

In particular, we will discuss

- series of numbers,
- series of functions, and
- power series.

The latter is more naturally expressed using a complex analysis framework, but we will present it, and important theorems for regular series, in the real analysis framework.

Outline

7.1 – Series of Numbers (p.3)

7.2 – Series of Functions (p.21)

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7.1 – Series of Numbers

Let $(x_n) \subseteq \mathbb{R}$. The **series** associated with (x_n) , denoted by

$$S_{(x_n)} = \sum_{n=1}^{\infty} x_n,$$

is the sequence (s_n) , where

$$s_1 = x_1$$

$$s_2 = x_1 + x_2$$

$$s_3 = x_1 + x_2 + x_3$$

...

If the sequence of partial sums s_n converges to S , we say the series $\sum_{n=1}^{\infty} x_n$ **converges to the sum S** .

We start by producing a necessary condition for convergence.

Theorem 70. *If $\sum_{n=1}^{\infty} x_n$ converges, then $x_n \rightarrow 0$.*

Proof.



We can bypass the need to know the limit in order to prove convergence.

Theorem 71. (CAUCHY CRITERION FOR SERIES)

The series $\sum_{n=1}^{\infty} x_n$ converges if and only if $\forall \varepsilon > 0, \exists N_\varepsilon \in \mathbb{N}$ such that

$$m > n > N_\varepsilon \implies |x_{n+1} + \cdots + x_m| < \varepsilon.$$

Proof.



But there are other tests that can be used to show the convergence of a series without knowing the limit.

Theorem 72. (COMPARISON TEST)

Let $\sum_{n=1}^{\infty} x_n$, $\sum_{n=1}^{\infty} y_n$ be series whose terms are all non-negative. If $\exists K \in \mathbb{N}$ such that $0 \leq x_n \leq y_n$ when $n > K$, then

1. $\sum_{n=1}^{\infty} y_n$ converges $\implies \sum_{n=1}^{\infty} x_n$ converges.

2. $\sum_{n=1}^{\infty} x_n$ diverges $\implies \sum_{n=1}^{\infty} y_n$ diverges.

Proof.



Examples: Discuss the convergence of 1. $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ and 2. $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

1.



2.



Theorem 73. (ALTERNATING SERIES TEST)

Let (a_n) be a sequence of non-negative numbers such that $a_n \searrow 0$ (i.e. $a_n \rightarrow 0$ and $a_{n+1} \leq a_n$). Then $\sum_{n=0}^{\infty} (-1)^n a_n$ converges.

Proof.



Even though it was not part of the statement of the Alternating Series Test, the proof allows us to conclude that the value of a convergent alternating series lies between a_{2k} and a_{2m+1} for all $k, m \in \mathbb{N}$.

Example: The alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges.

Proof.



Two other convergence tests are often used in practice: the Ratio Test and the Root Test. We shall prove only the Ratio Test, the proof for the Root Test is similar.

Theorem 74. (RATIO TEST)

Let (a_n) be a sequence of positive real numbers.

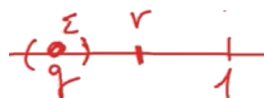
1. If $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} < 1$ then $\sum_{n=1}^{\infty} a_n$ converges.

2. If $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} > 1$ then $\sum_{n=1}^{\infty} a_n$ diverges.

If $\frac{a_{n+1}}{a_n} \rightarrow 1$, then the series may converge or diverge, depending on the nature of the terms a_n .

Proof.

1.



2.



The key parts of the proof (namely, the convergence of the tail in the first case and the condition $a_n \not\rightarrow 0$ in the second) are also valid if the statement is relaxed to some extent.

Theorem 74. (RATIO TEST – REPRISE)

Let (a_n) be a sequence of real numbers with $a_n \neq 0$ for all n .

1. If $\limsup_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$ then $\sum_{n=1}^{\infty} a_n$ converges.

2. If $\liminf_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| > 1$ then $\sum_{n=1}^{\infty} a_n$ diverges.

Theorem 75. (ROOT TEST)

Let (a_n) be a sequence of positive real numbers.

1. If $\limsup_{n \rightarrow \infty} \sqrt[n]{a_n} < 1$ then $\sum_{n=1}^{\infty} a_n$ converges.

2. If $\liminf_{n \rightarrow \infty} \sqrt[n]{a_n} > 1$ then $\sum_{n=1}^{\infty} a_n$ diverges.

If $\sqrt[n]{a_n} \rightarrow 1$, then the series may converge or diverge, depending on the nature of the terms a_n .

The proof of the Root Test follows the same general lines.

Examples: Discuss the convergence of $\sum_{n=1}^{\infty} \frac{(-1)^n}{n2^n}$; $\sum_{n=1}^{\infty} \frac{3^n}{n2^n}$; $\sum_{n=1}^{\infty} \frac{1}{n^p}$, $p > 0$.

1.



2.



3.

**Theorem 76.** (ABSOLUTE CONVERGENCE)

If the series $\sum_{n=0}^{\infty} |a_n|$ converges, so does $\sum_{n=0}^{\infty} a_n$ (not an “iff” statement).

Theorem 77. (SERIES REARRANGEMENT)

If the series $\sum_{n=0}^{\infty} |a_n|$ converges, so does $\sum_{n=0}^{\infty} a_{\varphi(n)}$, $\varphi : \mathbb{N} \rightarrow \mathbb{N}$ a bijection.

7.2 – Series of Functions

Series of functions play the same role for sequences of functions that series played for sequences of numbers.

Let $I \subseteq \mathbb{R}$ and $f_n : I \rightarrow \mathbb{R}, \forall n \in \mathbb{N}$. If the sequence of partial sums

$$s_1(x) = f_1(x)$$

$$s_2(x) = f_1(x) + f_2(x)$$

...

converges to some function $f : I \rightarrow \mathbb{R}$ for all $x \in I$, we say that the series of functions $\sum_{n=1}^{\infty} f_n$ **converges pointwise to f on I** .

Example: Consider the sequence of functions $f_n : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f_n(x) = x^n$ for each $n \in \mathbb{N}$. Does the sequence of partial sums $s_k(x)$ converge to some pointwise limit over some $A \subseteq \mathbb{R}$?

Solution.



If the sequence of partial sums (s_n) converges uniformly to f on I , we say that the series of functions $\sum_{n=1}^{\infty} f_n$ **converges uniformly to f on I .**

If the convergence of the series of functions is uniform, the limit interchange theorems can be applied.

Theorem 78. (CAUCHY CRITERION FOR SERIES OF FUNCTIONS)

Let $f_n : I \rightarrow \mathbb{R}$ for all $n \in \mathbb{N}$. The series of functions with term f_n converges uniformly to some function $f : I \rightarrow \mathbb{R}$ if and only if $\forall \varepsilon > 0$, $\exists N_\varepsilon \in \mathbb{N}$ (independent of $x \in I$) such that

$$m > n > N_\varepsilon \implies \left| \sum_{i=n+1}^m f_i(x) \right| < \varepsilon.$$

Proof.

The next result is a powerful tool to prove uniform convergence (and so to be able to use the Limit Interchange Theorems).

The simplicity of its proof belies its importance.

Theorem 79. (WEIERSTRASS M -TEST)

Let $f_n : I \rightarrow \mathbb{R}$ and $M_n \geq 0$ for all $n \in \mathbb{N}$. Assume that $|f_n(x)| \leq M_n$ for all $x \in I$, $n \in \mathbb{N}$. Then

$$\sum_{n=1}^{\infty} M_n \text{ converges} \implies \sum_{n=1}^{\infty} f_n \text{ converges uniformly on } I.$$

Proof.



Example: Let $\varepsilon \in (0, 1)$. Consider the sequence of functions $g_n : \mathbb{R} \rightarrow \mathbb{R}$ defined by $g_n(x) = nx^{n-1}$ for each $n \in \mathbb{N}$. Does $\sigma_k(x) \rightrightarrows \sigma(x)$ on $I_\varepsilon = (-1 + \varepsilon, 1 - \varepsilon)$ for some σ ? If so, find σ .

Solution.



Incidentally, Theorem 68 also tells us that $s_k(x) \Rightarrow \frac{1}{1-x}$ on I_ε , for all $0 < \varepsilon < 1$, and that for all $k \in \mathbb{N}$ and $x \in I_\varepsilon$, $\varepsilon \in (0, 1)$, we have

$$\sum_{n=0}^{\infty} \frac{d^k}{dx^k} [x^n] = \frac{d^k}{dx^k} \sum_{n=0}^{\infty} x^n = \frac{d^k}{dx^k} \left(\frac{1}{1-x} \right)$$

7.3 – Power Series

A **power series** around its **center** $x = x_0$ is a formal expression of the form

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

We have already seen an example of such a series, which converged uniformly on intervals containing $x_0 = 0$:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad \text{on } I_\varepsilon = (-1 + \varepsilon, 1 - \varepsilon), \quad \forall \varepsilon \in (0, 1)$$

(note, however, that the convergence is only pointwise on $(-1, 1)$).

Furthermore, the function $f : A \rightarrow \mathbb{R}$, $f(x) = \frac{1}{1-x}$ is defined for all $x \neq 1$, yet the power series $1+x+x^2+\dots$ does not converge to f outside of $(-1, 1)$.

Power series are commonly used as a formal guessing procedure to solve differential equations, but this is not a topic we will tackle at the moment.

A natural question to ask is: for which functions $f : A \rightarrow \mathbb{R}$ (and which A) can we find a sequence of coefficients (a_n) such that

$$f(x) = \sum_{n=0}^{\infty} a_n, \quad \forall x \in A?$$

Questions of this ilk are more naturally answered in \mathbb{C} ; a more complete treatment will be provided in a complex analysis course.

Examples: Where do the following power series converge?

$$1. \sum_{n=0}^{\infty} x^n, \quad 2. \sum_{n=1}^{\infty} (nx)^n, \quad 3. \sum_{n=1}^{\infty} \left(\frac{x}{n}\right)^n.$$

1. ■

2. ■

3.



The **radius of convergence** of a power series $\sum_{n=0}^{\infty} a_n(x - x_0)^n$ is

$$R = \frac{1}{\limsup_{n \rightarrow \infty} |a_n|^{1/n}}.$$

If the limit exists, we can replace \limsup by \lim . Intuitively, this says that for all large enough n ,

$$-R^{-n} \leq -|a_n| \leq a_n \leq |a_n| \leq R^{-n},$$

so that

$$-\sum_{n>N} \left(\frac{x-x_0}{R}\right)^n \leq \sum_{n>N} a_n(x-x_0)^n \leq \sum_{n>N} \left(\frac{x-x_0}{R}\right)^n.$$

The bounds are geometric series, and they converge when $|x-x_0| < R$.

We would expect the original power series to converge on the **interval of convergence** $|x-x_0| < R$.

Theorem 80. *Let R be the radius of convergence of the power series*

$$\sum_{n=0}^{\infty} a_n(x - x_0)^n.$$

Then, if

- *$R = 0$, the power series converges for $x = x_0$ and diverges for $x \neq x_0$;*
- *$R = \infty$, the power series converges absolutely on \mathbb{R} , and*
- *$0 < R < \infty$, the power series converges absolutely on $|x - x_0| < R$, diverges on $|x - x_0| > R$; the extremities must be analyzed separately.*

Proof. ■

Theorem 81. *The power series of Theorem 80 converges uniformly on any compact sub-interval*

$$[a, b] \subseteq (x_0 - R, x_0 + R).$$

Proof.



In what follows, we let $f : (x_0 - R, x_0 + R) \rightarrow \mathbb{R}$ be the function defined by

$$f(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n, \quad \text{and} \quad s_N(x) = \sum_{n=0}^N a_n(x - x_0)^n.$$

Theorem 82. *The function f is continuous on any closed bounded interval $[a, b] \subseteq (x_0 - R, x_0 + R)$.*

Proof.



Theorem 83. *Let $x \in (x_0 - R, x_0 + R)$. Then f is Riemann-integrable between x_0 and x and*

$$\int_{x_0}^x f(t) dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1} (x - x_0)^{n+1}.$$

Proof.



Theorem 84. *The function f is differentiable on $(x_0 - R, x_0 + R)$ and*

$$f'(x) = \sum_{n=1}^{\infty} n a_n (x - x_0)^{n-1}.$$

Proof.



How do we compute the power series coefficients a_n ? Combining Theorems 82 and 84, we see that f is **smooth** in its interval of convergence (i.e. all of its derivatives are continuous).

Theorem 85. *If $R > 0$, then*

$$a_n = \frac{f^{(n)}(x_0)}{n!}.$$

Proof.



Corollary. If $\exists r > 0$ such that

$$f(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n \quad \text{and} \quad g(x) = \sum_{n=0}^{\infty} b_n(x - x_0)^n$$

and $f(x) = g(x)$ for all $x \in (x_0 - r, x_0 + r)$, then $a_n = b_n$ for all $n \in \mathbb{N}$.

Attempts to strengthen this uniqueness result must fail.

Example: Consider the function $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$f(x) = \begin{cases} \exp(-1/x^2), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Show that f does not have a power series expansion.

Proof.



Thus, we cannot always assume that a function is equal to its power series.

There are other ways to expand a function as infinite series, most notable being **Laurent Series** and **Fourier Series**. These topics are covered in courses in complex analysis and partial differential equations, respectively.

7.4 – Exercises

1. Answer the following questions about series.

(a) If $\sum_{k=1}^{\infty} (a_k + b_k)$ converges, what about $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$?

(b) If $\sum_{k=1}^{\infty} (a_k + b_k)$ diverges, what about $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$?

(c) If $\sum_{k=1}^{\infty} (a_{2k} + a_{2k-1})$ converges, what about $\sum_{k=1}^{\infty} a_k$?

(d) If $\sum_{k=1}^{\infty} a_k$ converges, what about $\sum_{k=1}^{\infty} (a_{2k} + a_{2k-1})$?

2. Show that

$$\frac{1}{r-1} = \frac{1}{r+1} + \frac{2}{r^2+1} + \frac{4}{r^4+1} + \frac{8}{r^8+1} + \dots$$

for all $r > 1$.

3. Using Riemann integration, find the values of p for which the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges (compare with the approach used in the notes).

4. Which of the following series converge?

(a) $\sum_{n=1}^{\infty} \frac{n(n+1)}{(n+2)^2}$

(b) $\sum_{n=1}^{\infty} \frac{2 + \sin^3(n+1)}{2^n + n^2}$

(c) $\sum_{n=1}^{\infty} \frac{1}{2^n - 1 + \cos^2 n^3}$

(d) $\sum_{n=1}^{\infty} \frac{n+1}{n^2+1}$

(e) $\sum_{n=1}^{\infty} \frac{n+1}{n^3+1}$

$$(f) \sum_{n=1}^{\infty} \frac{n!}{n^n}$$

$$(g) \sum_{n=1}^{\infty} \frac{n!}{5^n}$$

$$(h) \sum_{n=1}^{\infty} \frac{n^n}{3^{1+2n}}$$

$$(i) \sum_{n=1}^{\infty} \left(\frac{5n + 3n^3}{7n^3 + 2} \right)^n$$

5. Give an example of a power series $\sum_{k=0}^{\infty} a_k x^k$ with interval of convergence $[-\sqrt{2}, \sqrt{2})$.

6. Find the values of x for which the following series converge:

$$(a) \sum_{n=1}^{\infty} (nx)^n;$$

$$(b) \sum_{n=1}^{\infty} x^n;$$

$$(c) \sum_{n=1}^{\infty} \frac{x^n}{n^2};$$

$$(d) \sum_{n=1}^{\infty} \frac{x^n}{n!}.$$

7. If the power series $\sum a_k x^k$ has radius of convergence R , what is the radius of convergence of the series $\sum a_k x^{2k}$?

8. Obtain power series expansions for the following functions.

$$(a) \frac{x}{1+x^2};$$

$$(b) \frac{x}{(1+x^2)^2};$$

$$(c) \frac{x}{1+x^3};$$

$$(d) \frac{x^2}{1+x^3};$$

$$(e) f(x) = \int_0^1 \frac{1 - e^{-sx}}{s} ds, \text{ about } x = 0.$$

Solutions

1. Proof.

2. **Proof.**

3. Proof.

4. Proof.

5. Proof.

6. Proof.

7. Proof.

8. Proof.

