MAT 2125 – Final Exam

You must provide *complete*, *clear* and *precise* solutions to the questions to score full marks. The value of each question is indicated at the start of the question.

1. **True or False:** determine the veracity of the following statements. If false, provide a counterexample. [1 mark each]

   i. If $S_1, S_2, \ldots, S_n$ are countable sets, then $S = \bigcup_{i=1}^{n} S_n$ is countable as well.
   Answer: True

   ii. A sequence is bounded if and only if all subsequences are bounded.
   Answer: True

   iii. If $\lim_{n\to\infty} x_n$ and $\lim_{n\to\infty} y_n$ are real numbers, then $\lim_{n\to\infty} \frac{x_n}{y_n}$ exists as well.
   Answer: False; $x_n \equiv 1$, $y_n \equiv 0$.y

   iv. If $\lim_{n\to\infty} x_n$ and $\lim_{n\to\infty} y_n$ are real numbers, then $\lim_{n\to\infty} x_n y_n$ exists as well.
   Answer: True

   v. If (x_n) is unbounded, then $\left(\frac{1}{x_n}\right)$ is bounded.
   Answer: False; $x_{2n} = 2^n$, $x_{2n-1} = 2^{-n}$ for $n \in \mathbb{N}$.

   vi. If $S_1, S_2, \ldots$ are compact sets, then $S = \bigcup_{n=1}^{\infty} S_n$ is compact as well.
   Answer: False; $S_n = [-n, n]$ is compact in $\mathbb{R}$ for all $n \in \mathbb{N}$, but $S = \mathbb{R}$ is not compact.

   vii. A function defined on an unbounded set cannot be uniformly continuous.
   Answer: False; $f : \mathbb{R} \to \mathbb{R}$, $f \equiv 0$ is uniformly continuous.

   viii. If f is differentiable at c, then f is continuous at c.
   Answer: True

   ix. If A is open in $\mathbb{R}$ and $f : A \to \mathbb{R}$ is continuous, then $f(A) = \{f(a) \mid a \in A\}$ is open in $\mathbb{R}$.
   Answer: False; $A = \mathbb{R}$, $f(x) \equiv 0$.

   x. If $(a_n)_n, (b_n)_n$ satisfy $|a_n| \leq |b_n|$ for all $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n} a_n$ converges.
   Answer: False; $a_n = \frac{1}{n}$, $b_n = (-1)^n a_n$.

   xi. Every bounded function $f : [a, b] \to \mathbb{R}$ is Riemann-integrable.
   Answer: False; $f(x) = \chi_{\mathbb{Q}}(x)$.

   xii. If (f_n) is a sequence of Riemann-integrable functions on $[a, b]$ and f_n converges uniformly to f on $[a, b]$, then f is Riemann-integrable on $[a, b]$.
   Answer: True.

   xiii. If f_n converges uniformly to f on $[a, a + 1]$ for all $a \in \mathbb{Z}$, then f_n converges uniformly to f on $\mathbb{R}$.
   Answer: False; $(x + \frac{1}{n})^2 \rightrightarrows x^2$ on $[a, a + 1]$ for all $a \in \mathbb{Z}$, but $(x + \frac{1}{n})^2 \not\rightrightarrows x^2$ on $\mathbb{R}$.

   xiv. Let $f : \mathbb{R} \to \mathbb{R}$ be infinitely differentiable, and define $a_n = f^{(n)}(0)$. If $\sum_{n=1}^{\infty} \frac{a_n}{n!} x^n$ converges uniformly to a function g on an interval $[-C, C]$, then $f(x) = g(x)$ for all $x \in (-C, C)$.
   Answer: False; $f(x) = e^{-\frac{1}{x^2}}$ for $x \neq 0$, $f(0) = 0$.

xv. Let $f_n : [0,1] \to \mathbb{R}$, $n \in \mathbb{N}$ be a sequence of continuous functions that converges uniformly to f on $[0,1]$. Then f is uniformly continuous on $[0,1]$.
Answer: True

2. **Examples:** For each set of conditions, provide an example satisfying the conditions. It should be easy to see by inspection that the examples are correct. [1 mark each]

   i. A set $A \subseteq \mathbb{R}$ that has a finite supremum but not a maximum.
   Answer: $A = (0,1)$.

   ii. A sequence that has no convergent subsequence.
   Answer: $x_n = 2^n$.

   iii. A function $f : (0,1) \to \mathbb{R}$ that is continuous but not uniformly continuous.
   Answer: $f(x) = x^{-1}$.

   iv. A sequence of functions $f_n : (0,1) \to \mathbb{R}$ that converge to some function f, but not uniformly.
   Answer: $f_n(x) = x^n$, $f \equiv 0$.

   v. A point $y \in [0,1]$ and a function $f : [0,1] \to \mathbb{R}$ so that $F(x) = \int_0^x f$ exists for all $x \in [0,1]$, but with at least one $y \in [0,1]$ such that $F'(y) \neq f(y)$.
   Answer: $f(x) = \chi_{(0.5,1]}(x)$, $y = 0.5$.

3. **Short Proofs:** provide a proof for 3 of the following statements. [3 marks each]

   i. Using only the field axioms, the order axioms, and/or the Archimedean Property, show that if u, x, y are real numbers such that $u > 0$ and $x < y$, there exists $r \in \mathbb{Q}$ such that $x < ru < y$.

   Proof: Note that $y > x \implies y - x > 0 \implies \frac{1}{y-x} > 0$. By the Archimedean Property, $\exists n, m \in \mathbb{N}^\times$ such that

$$n > \frac{1}{y-x} > 0 \quad \text{and} \quad m - 1 \leq nx < m.$$

   Since $n(y-x) > 1$, then

$$ny - nx > 1 \implies ny - 1 > nx.$$

   By transitivity of the order,

$$ny - 1 > nx \geq m - 1 \implies ny > m.$$

   But $m > nx$, so since $n > 0$, then $y > \frac{m}{n} > x$. Select $r = \frac{m}{n}$. ■

   ii. Let (a_n) be a sequence. If $\lim_{n\to\infty} a_{3n} = \lim_{n\to\infty} a_{3n+1} = \lim_{n\to\infty} a_{3n+2} = L$, show that $\lim_{n\to\infty} a_n = L$.

   Proof: Let $\varepsilon > 0$. Then $\exists N_0, N_1, N_2 \in \mathbb{N}$ such that

$$n > N_0 \implies |a_{3n} - L| < \varepsilon, \quad n > N_1 \implies |a_{3n+1} - L| < \varepsilon, \quad n > N_2 \implies |a_{3n+2} - L| < \varepsilon$$

   Set $K_\varepsilon = 3\max\{N_0, N_1, N_2\} + 2$. Then

$$k > K_\varepsilon \implies |a_k - L| < \varepsilon,$$

   which is to say that $a_k \to L$. ■

iii. Suppose the sequence $(\mathbf{x}_n)$ in $\mathbb{R}^d$ converges to $\mathbf{x}$. Show that, for any norm $\|\cdot\|$, the sequence $(\|\mathbf{x}_n\|)$ converges to $\|\mathbf{x}\|$.

Proof: Let $\|\cdot\|$ be a norm on $\mathbb{R}^d$ and let $\varepsilon > 0$. Since $\mathbf{x}_n \to \mathbf{x}$, $\exists N_\varepsilon \in \mathbb{N}$ such that

$$n > N_\varepsilon \implies \|\mathbf{x}_n - \mathbf{x}\| < \varepsilon.$$

But for any norm we have $|\|\mathbf{x}_n\| - \|\mathbf{x}\|| \leq \|\mathbf{x}_n - \mathbf{x}\|$, so

$$n > N_\varepsilon \implies |\|\mathbf{x}_n\| - \|\mathbf{x}\|| \leq \|\mathbf{x}_n - \mathbf{x}\| < \varepsilon,$$

which is to say $\|\mathbf{x}_n| \to \|\mathbf{x}\|$. ■

iv. Let $f, g : [0, \infty) \to \mathbb{R}$ be differentiable functions such that $g'(x) \geq f'(x)$ for all $x \geq 0$. Give a condition on $g(0)$ and $f(0)$ (with proof) that will guarantee that $g(x) \geq f(x)$ for all $x \geq 0$.

Proof: Consider the function $h : [0, \infty) \to \mathbb{R}$ defined by

$$h(x) = g(x) - f(x), \quad \text{for all } x \in [0, \infty).$$

As f and g are both differentiable on $[0, \infty)$, so is h, and

$$h'(x) = g'(x) - f'(x) \geq 0, \quad \text{for all } x \in [0, \infty).$$

Thus h is increasing on $[0, \infty)$, so that $h(x) \geq h(0)$ for all $x \in [0, \infty)$.

If furthermore $h(x) \geq 0$ for all $x \in [0, \infty)$, then $g(x) - f(x) \geq 0 \implies g(x) \geq f(x)$ for all $x \in [0, \infty)$. It is thus sufficient to have $h(0) \geq 0$, which is to say, $g(0) \geq f(0)$. ■

v. If (f_n) is a sequence of continuous functions on A such that f_n converges uniformly to $\mathbb{R}$ is continuous.

Proof: Let $\varepsilon > 0$. By definition, $\exists H_{\varepsilon/3} \in \mathbb{N}$ such that $n > H_{\varepsilon/3}$ and $x \in A \implies |f_n(x) - f(x)| < \frac{\varepsilon}{3}$. Let $c \in A$. According to the Triangle Inequality,

$$\begin{aligned} n > H_{\varepsilon/3} \implies |f(x) - f(c)| &\leq |f(x) - f_{H_{\varepsilon/3}}(x)| + |f_{H_{\varepsilon/3}}(x) - f_{H_{\varepsilon/3}}(c)| + |f_{H_{\varepsilon/3}}(c) - f(c)| \\ &< \tfrac{\varepsilon}{3} + |f_{H_{\varepsilon/3}}(x) - f_{H_{\varepsilon/3}}(c)| + \tfrac{\varepsilon}{3} \end{aligned}$$

But $f_{H_{\varepsilon/3}}$ is continuous at c, so $\exists \delta_{\varepsilon/3} > 0$ such that $|f_{H_{\varepsilon/3}}(x) - f_{H_{\varepsilon/3}}(c)| < \frac{\varepsilon}{3}$ when $x \in A$ and $|x - c| < \delta_{\varepsilon/3}$. Thus $|f(x) - f(c)| < \varepsilon$ whenever $x \in A$ and $|x - c| < \delta_{\varepsilon/3}$, so f is continuous at c. As $c \in A$ is arbitrary, f is continuous on A. ■

vi. Let $f : \mathbb{R} \to \mathbb{R}$ be such that f is differentiable at least 15 times on $\mathbb{R}$. Assume further that there exist 20 distinct points $x_1 < x_2 < \cdots < x_{19} < x_{20}$ with $f(x_i) = 1$, $1 \leq i \leq 20$. Show that there exists some point x so that the tenth derivative of f satisfies $f^{(10)}(x) = 0$.

Proof: Since f is at least 15 times differentiable on $\mathbb{R}$, then $f, f', f'', \ldots, f^{(14)}$ are continuous on $\mathbb{R}$. By the Mean Value Theorem, for all $1 \leq i \leq 19$, $\exists x_i' \in (x_i, x_{i+1})$ such that

$$f'(x_i') = \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i} = 0.$$

For every $1 \le i \le 19$, f' is continuous on $[x'_i, x'_{i+1}]$, differentiable on (x'_i, x'_{i+1}), and $f'(x'_i) = f'(x'_{i+1})$; consequently, $\exists x''_i \in (x'_i, x'_{i+1})$ such that $f''(x''_i) = 0$, according to Rolle's Theorem.

For every $1 \le i \le 18$, f'' is continuous on $[x''_i, x''_{i+1}]$, differentiable on (x''_i, x''_{i+1}), and $f''(x''_i) = f''(x''_{i+1})$; consequently, $\exists x'''_i \in (x''_i, x''_{i+1})$ such that $f'''(x'''_i) = 0$, according to Rolle's Theorem.

Continuing on this way, we see that for every $1 \le i \le 11$, $f^{(9)}$ is continuous on $[x_i^{(9)}, x_{i+1}^{(9)}]$, differentiable on $(x_i^{(9)}, x_{i+1}^{(9)})$, and $f^{(9)}(x_i^{(9)}) = x_{i+1}^{(9)}$; consequently, $\exists x_i^{(10)} \in (x_i^{(9)}, x_{i+1}^{(9)})$ such that $f^{(10)}(x_i^{(10)}) = 0$, according to Rolle's Theorem. Set $x = x_1^{(10)}$. This completes the proof. ■

4. **Computations and Applications:** answer 3 of the following questions. [3 marks each]

   i. Let $f : [-1, 1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x^4 \sin\left(\frac{1}{x^4}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Show f is differentiable on $[-1, 1]$ and find f'. Is f' continuous on $[-1, 1]$?

Proof: If $x \neq 0$, then $f(x)$ is differentiable, being the product of two differentiable functions, and

$$f'(x) = 4x^3 \sin\left(\tfrac{1}{x^4}\right) - 4x \cos\left(\tfrac{1}{x^4}\right).$$

If $x = 0$, then $f'(0) = 0$. Indeed, $\forall \varepsilon > 0$, set $\delta_\varepsilon = \varepsilon^{1/3}$. Then for $x \in [-1, 1]$,

$$0 < |x| < \delta_\varepsilon \implies \left| \frac{x^4 \sin\left(\frac{1}{x^4}\right) - 0}{x - 0} - 0 \right| = |x^3 \sin\left(\tfrac{1}{x^4}\right)| \le |x|^3 < \delta_\varepsilon^3 = \varepsilon,$$

and so

$$f'(x) = \begin{cases} 4x^3 \sin\left(\frac{1}{x^4}\right) - 4x \cos\left(\frac{1}{x^4}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

The derivative f' is continuous at all $x \in [-1, 1] \setminus \{0\}$, being the sum and product of continuous functions there. But $\lim_{x \to 0} f'(x)$ does not exist (and so is not equal to $f'(0)$): indeed, let $x_n = (\frac{1}{n\pi})^{1/4}$. Then $x_n \to 0$, $x_n \neq 0$, and the sequence

$$f'(x_n) = 4(\tfrac{1}{n\pi})^{1/3} \sin(n\pi) - 4(n\pi)^{1/4} \cos(n\pi) = -4(n\pi)^{1/4}(-1)^n$$

diverges as it is not bounded. Thus f' is not continuous on $[-1, 1]$. ■

   ii. Let $f : [0, 2] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1, & x \in [0, 1) \\ 3, & x = 1 \\ -3, & x \in (1, 2] \end{cases}$$

Show directly that f is Riemann-integrable on $[0, 2]$ and compute $\int_0^2 f$.

Proof: Let $\varepsilon > 0$ and define the partition $P_\varepsilon = \{0, 1-\varepsilon, 1+\varepsilon, 2\}$. Since f is bounded on $[0,2]$, $L(f) \leq U(f)$ exist and

$$L(f) \geq L(P_\varepsilon; f) = 1 \cdot (1-\varepsilon) + (-3) \cdot (2\varepsilon) + (-3) \cdot (1-\varepsilon) = -2 - 4\varepsilon, \quad \text{and}$$
$$U(f) \leq U(P_\varepsilon; f) = 1 \cdot (1-\varepsilon) + (3) \cdot (2\varepsilon) + (-3) \cdot (1-\varepsilon) = -2 + 2\varepsilon.$$

Hence

$$-2 - 4\varepsilon \leq L(f) \leq U(f) \leq -2 + 2\varepsilon, \quad \text{for all } \varepsilon > 0.$$

Since $\varepsilon > 0$ is arbitrary, then $-2 \leq L(f) \leq U(f) \leq -2$; by definition, f is Riemann-integrable on $[0,2]$ and $L(f) = U(f) = \int_0^2 f = -2$. ■

iii. Assume that f is continuous at 0. Show that $g(x) \equiv \sin^2(x) f(x)$ is differentiable at 0.

Proof: Let $M = \sup_{|x| \leq 1} |f(x)|$; by continuity of f we know $M < \infty$. Then

$$\limsup_{x \to 0} \left| \frac{\sin^2(x) f(x) - \sin^2(0) f(0)}{x - 0} \right| = \limsup_{x \to 0} \left| \frac{\sin^2(x) f(x)}{x} \right| \leq M \cdot \lim_{x \to 0} \left| \frac{\sin^2(x)}{x} \right| = 0,$$

so $g'(0) = 0$. ■

iv. Show that $\sum_{k=0}^{\infty} (k^2 + k + 1) \sin(kx) x^k$ converges uniformly on $x \in [-a, a]$ for all $0 < a < 1$.

Proof: Let $a \in (0,1)$. Consider the functions $f_k : [-a, a] \to \mathbb{R}$, $x \mapsto (k^2+k+1)\sin(kx)x^k$, $k \geq 1$. We have

$$|f_k(x)| \leq M_k = (k^2 + k + 1) \cdot a^k, \quad \text{for all } x \in [-a, a].$$

Then $M_k > 0$ for all $k \in \mathbb{N}$. If $\sum_{k=0}^{\infty} M_k$ converges, $\sum_{k=0}^{\infty} f_k(x)$ converges uniformly on $[-a, a]$, according to the Weierstrass $M-$Test. But

$$\lim_{k \to \infty} \left| \frac{M_{k+1}}{M_k} \right| = \lim_{k \to \infty} \left| \frac{((k+1)^2 + (k+1) + 1) \cdot a^{k+1}}{(k^2 + k + 1) \cdot a^k} \right| = a \cdot \lim_{k \to \infty} \frac{k^2 + 3k + 3}{k^2 + k + 1} = a < 1.$$

According to the Ratio Test, $\sum_k M_k$ converges, so $\sum f_k$ converges uniformly on $[-a, a]$. ■

v. Let $f(x) = \sum_{k=0}^{\infty} \frac{(3x)^k}{k!}$. Show that $f'(x)$ exists and satisfies $f'(x) = 3f(x)$ over some interval of convergence $(-R, R)$, without directly using the fact that $\frac{d}{dx} e^{Cx} = Ce^{Cx}$.

Proof: Let $a_k = \frac{3^k}{k!}$, $k \in \mathbb{N}$. The radius of convergence of the power series is

$$R = \limsup_{k \to \infty} \left| \frac{a_k}{a_{k+1}} \right| = \limsup_{k \to \infty} \frac{3^k}{k!} \cdot \frac{(k+1)!}{3^{k+1}} = \limsup_{k \to \infty} \frac{k+1}{3} = \infty,$$

and so the power series converges uniformly on the convergence interval $\mathbb{R} = (-\infty, \infty)$. According to the Limit Interchange Theorem for power series, we thus have

$$f'(x) = \frac{d}{dx} \sum_{k=0}^{\infty} \frac{(3x)^k}{k!} = \sum_{k=0}^{\infty} \frac{d}{dx} \frac{(3x)^k}{k!} = \sum_{k=0}^{\infty} \frac{k(3x)^{k-1} \cdot 3}{k!} = \sum_{k=1}^{\infty} 3 \frac{(3x)^{k-1} 3}{(k-1)!} = 3f(x),$$

which incidentally also shows that f' exists. ■

vi. Show that $\lim_{n\to\infty} \int_5^{10} x^{12} \sin^{19}(\frac{x}{n})\, dx = 0.$

Proof: Let $n \in \mathbb{N}$. Then $f_n : [5,10] \to \mathbb{R}$, defined by $f_n(x) = x^{12}\sin^{19}(\frac{x}{n})$, is Riemann-integrable on $[5,10]$ since it is continuous on $[5,10]$. However $x^{12}\sin^{19}(\frac{x}{n}) \rightrightarrows 0$ on $[5,10]$. Indeed, let $\varepsilon > 0$, then $\exists N_\varepsilon > \sqrt[19]{\frac{10^{31}}{\varepsilon}}$ (independent of x) such that

$$n > N_\varepsilon \text{ and } x \in [5,10] \implies \left|x^{12}\sin^{19}(\tfrac{x}{n}) - 0\right| = x^{12}\left|\sin^{19}(\tfrac{x}{n}) - 0\right| \le 10^{12}\left|\frac{x}{n}\right|^{19} < \frac{|10|^{31}}{N_\varepsilon^{19}} < \varepsilon.$$

Thus,

$$\lim_{n\to\infty} \int_5^{10} x^{12}\sin^{19}(\tfrac{x}{n})\, dx = \int_5^{10} \lim_{n\to\infty} x^{12}\sin^{19}(\tfrac{x}{n})\, dx = \int_5^{10} 0\, dx = 0,$$

according to the Limit Interchange Theorem for integrals. ■

5. **Longer Proofs:** answer all questions in this section. [3 marks each]

i. Let $f : [a,b] \to \mathbb{R}$ be differentiable on $[a,b]$ and $(x_n) \subseteq [a,b]$ be a sequence such that $x_n \to p$. If all the x_n are distinct and $f(x_n) = 0$ for all $n \in \mathbb{N}$, show $f(p) = 0$ and $f'(p) = 0$.

Proof: f is differentiable on $[a,b]$ and so is continuous on $[a,b]$, so

$$0 = \lim_{n\to\infty} f(x_n) = f\big(\lim_{n\to\infty} x_n\big) = f(p).$$

Since the derivative of f at p exists and since $x_n \neq p$ for all $n \in \mathbb{N}$, we can use the sequential definition to compute the derivative at p:

$$f'(p) = \lim_{n\to\infty} \frac{f(x_n) - f(p)}{x_n - p} = \lim_{n\to\infty} \frac{0-0}{x_n - p} = 0.$$

This completes the proof. ■

ii. Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ be given by a power series that converges uniformly on an interval $[-C, C]$, and for which all terms (a_n) are nonzero. Show that there exists some $\delta > 0$ so that f is monotone on $[-\delta, \delta]$.

Proof: Since the power series converges uniformly, we have

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

for $x \in (-C, C)$. By assumption, $|a_1| > 0$; WLOG, assume that $a_1 > 0$. We then have, for $x \in (-C, C)$,

$$\begin{aligned} |f'(x) - a_1| &\le \sum_{n=1}^{\infty} (n+1)|a_n|\, |x|^n = |x| \sum_{n=1}^{\infty} (n+1)|a_n|\, |x|^{n-1} \\ &\le |x| \sum_{n=1}^{\infty} (n+1)|a_n|\, (0.1C)^{n-1} \equiv |x| A. \end{aligned}$$

Note that $A < \infty$, since power series are uniformly and absolutely convergent on open intervals within their radii of convergence. Thus, for all $|x| < \frac{|a_1|}{2A}$,

$$f'(x) \geq a_1 - A|x| \geq \frac{a_1}{2} > 0,$$

so f' is monotone on $(\frac{-|a_1|}{2A}, \frac{|a_1|}{2A})$. ■

iii. Define the **closure** $\overline{A}$ of a set $A \subseteq \mathbb{R}^d$ to be the union of A and the boundary ∂A of A. Show that:

(a) $\overline{A}$ is closed.

Answer: Let $\mathbf{x} \notin \overline{A}$. Since $\mathbf{x}$ is not in the boundary, there exists $r > 0$ s.t. one of $B_{3r}(\mathbf{x}) \cap A$, $B_{3r}(\mathbf{x}) \cap A^c$ is empty. But $\mathbf{x} \notin A$, so $B_{3r}(\mathbf{x}) \cap A^c \neq \varnothing$, and we conclude $B_{3r}(\mathbf{x}) \cap A = \varnothing$.

But $B_r(\mathbf{x}) \cap \overline{A} = \varnothing$. Indeed, if not, there exists $\mathbf{y} \in B_r(\mathbf{x}) \cap \overline{A}$; by the last paragraph $\mathbf{y} \notin A$ but is in the boundary of A. But then $B_r(\mathbf{y}) \cap A \subseteq B_{3r}(x) \cap A = \varnothing$, so in fact $\mathbf{y}$ is not in boundary. ■

(b) If $A \subseteq \mathbb{R}^d$ is bounded, then $\overline{A}$ is bounded.

Proof: Assume $\sup_{\mathbf{a} \in A} \|\mathbf{a}\| = M < \infty$. Let $\mathbf{b} \in \overline{A}$ but $\mathbf{b} \notin A$. Then $B_1(\mathbf{b}) \cap A$ contains a point $\mathbf{c}$, and so $\|\mathbf{b}\| \leq \|\mathbf{c}\| + 1 \leq M + 1$. Thus, in both cases $\mathbf{a} \in A$, or $\mathbf{a} \in \overline{A}$ but $\mathbf{a} \notin A$, we have $\|\mathbf{a}\| \leq M + 1 < \infty$. ■

(c) Fix $\varepsilon > 0$ and a bounded set $A \subseteq \mathbb{R}^d$. Show that there exists a finite collection of points $\mathbf{x}_1, \ldots, \mathbf{x}_n \in \mathbb{R}^d$ so that

$$A \subseteq \bigcup_{i=1}^{n} B_\varepsilon(\mathbf{x}_i),$$

where $B_\varepsilon(\mathbf{x}) = \{\mathbf{y} \in \mathbb{R}^d : \|\mathbf{x} - \mathbf{y}\| < \varepsilon\}$ is the usual open ball of radius ε around $\mathbf{x}$.

Proof: By the previous two parts, $\overline{A}$ is closed and bounded (hence compact). Consider the collection of sets $\{B_\varepsilon(\mathbf{a})\}_{\mathbf{a} \in \overline{A}}$. This is clearly an open cover of $\overline{A}$, so it has a finite subcover $\{B_\varepsilon(\mathbf{a}_i)\}_{i=1}^n$. Thus

$$A \subseteq \overline{A} \subseteq \bigcup_{i=1}^{n} B_\varepsilon(\mathbf{a}_i).$$

This concludes the proof. ■