# Mathematical Analysis

# Chapter 10
# Metric Spaces and Topology

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# Overview

One of the natural ways we can extend the concepts we have discussed in the previous chapters is by moving from $\mathbb{R}$ to $\mathbb{R}^m$.

Some of the notions that generalize nicely to vectors and functions on vectors include compactness and connectedness.

**Notation:** The symbol $\mathbb{K}$ is sometimes used to denote either $\mathbb{R}$ or $\mathbb{C}$.

$C_{\mathbb{R}}([0,1])$ represents the $\mathbb{R}-$vector space of continuous functions $[0,1] \to \mathbb{R}$.

$\mathcal{F}_{\mathbb{R}}([0,1])$ represents the $\mathbb{R}-$vector space of functions $[0,1] \to \mathbb{R}$.

# Outline

# 10.1 – Compact Spaces

Let $A$ be a finite set. A function $f : A \to \mathbb{K}$ is necessarily **bounded** (in the sense that $\exists M \in \mathbb{K}$ such that $|f(a)| \leq M$ for all $a \in A$).

Might this be due to the **finiteness** of $A$? While finiteness is sufficient, it is not a necessary condition for boundedness: the function $\chi_{\mathbb{Q}} : [0,1] \to \mathbb{R}$ is bounded, even though its domain is the infinite set $[0,1]$.

Might it be due to the **boundedness of the domain** of the function? This is neither sufficient nor necessary, as can be seen from the functions

$$f : [0,1] \to \mathbb{R}, \quad f(x) = \frac{1}{x} \text{ for } x > 0, \text{ and } f(0) = 0,$$

and $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = \exp(-x^2)$.

Might it be due to the **continuous nature** of the function? We have examples of continuous function being bounded, others being unbounded; and non-continuous functions being bounded, others being unbounded.

A condition on the domain of the function alone cannot guarantee boundedness; and neither can one on the nature of the function.

However, a **combination** of two conditions, one each on the domain and on the function, can provide such a guarantee.

In this section, we study the appropriate property on the domain, that of **compactness**, which generalizes the property of finiteness.

The definition is due to Borel and Lebesgue, and is applicable to metric and general topological spaces alike.

# 10.1.1 – The Borel-Lebesgue Property

A space $E$ is **compact** if any family of open subsets covering $E$ contains a finite sub-family which also covers $E$.

In other words, $E$ is compact if, for any collection $\mathcal{U} = \{U_i\}_{i\in I}$ of open subsets $U_i \subseteq_O E$ with $E \subseteq \bigcup_{i\in I} U_i$, $\exists$ a finite $J \subseteq I$ s.t. $E \subseteq \bigcup_{j\in J} U_j$.

**Examples:**

1. Every finite metric space $(E, d)$ is compact.

   **Proof.**

   ■

2. In the standard topology, $\mathbb{R}$ is not compact.

   **Proof.**

   ■

3. Show that $\mathbb{R}$ is compact in the indiscrete topology.

   **Proof.**

   ■

4. Show that any compact metric $(E, d)$ space is bounded.

   **Proof.**

   $\blacksquare$

By abuse of notation, we will often write: "let $\bigcup U_i$ be an open cover of $E$" rather than "let $\{U_i\}$ be an open cover of $E$," as in the examples above.

Incidentally, does the fourth example contradict the third one? What does that imply about the indiscrete topology?

The duality open/closed, union/intersection yields an equivalent definition: a space $E$ is **compact** if any family of closed subsets of $E$ with an empty intersection contains a finite sub-family whose intersection is also empty.

In other words, $E$ is compact if, for any collection $\mathcal{W} = \{V_i\}_{i\in I}$ of closed subsets $V_i \subseteq_C E$ with $\bigcap_{i\in I} V_i = \varnothing$, $\exists$ a finite $J \subseteq I$ s.t. $\bigcap_{j\in J} V_j = \varnothing$.

**Proposition 115.** *Let $(F_n)_{n\geq 1}$ be a decreasing sequence of non-empty closed subsets of a compact space $E$. Then $\bigcap_{n\geq 1} F_n \neq \varnothing$.*

**Proof.**

∎

Continuous functions on compact domains have quite useful properties.

**Proposition 116.** *Let* $f : (E, d) \to (F, \delta)$ *be any continuous function over a compact metric space. Then* $f$ *is uniformly continuous.*

**Proof.**

$\blacksquare$

A subset $A \subseteq E$ is deemed to be a **compact subset of** $E$, which we denote by $A \subseteq_K E$, if any family of open subsets of $E$ covering $A$ contains a finite sub-family which also covers $A$.

**Proposition 117.** *A finite union of compact subsets of $E$ is itself compact.*

**Proof.**

$\blacksquare$

The infinite union of compact subsets could be compact or not.

**Examples**:

1.

2.

3.

# 10.1.2 – The Bolzano-Weierstrass Property

For metric spaces, compactness can also be established via a property of **sequences** which is often easier to ascertain than the Borel-Lebesgue property – ⚠ **the two properties are not equivalent in general for non-metric spaces**.

Let $(E, d)$ be a metric space. We say that $E$ is **precompact** if $\forall \varepsilon > 0$, $\exists \mathbf{x}_1, \ldots, \mathbf{x}_n \in E$ such that $E = \bigcup_{i=1}^{n} B(\mathbf{x}_i, \varepsilon)$.

**Proposition 118.** *A compact space is precompact.*

**Proof.** ■

**Theorem 119.** *Let $(E, d)$ be a metric space. Then $E$ is compact if and only if any sequence in $E$ has a convergent sub-sequence in $E$.*

# Proof.

■

The following result has a similar flavour.

**Theorem 120.** *Let $(E, d)$ be a metric space. Then $E$ is compact if and only if any sequence in $E$ has a limit point if and only if every infinite subset of $E$ has a cluster point.*

**Proof.** ■

It is typically easier to show that the Bolzano-Weierstrass is violated than to show that it holds.

**Example:** Show that the set $(0, 1)$ is not a compact subset of $\mathbb{R}$ in the usual topology.

**Proof.**

■

Compact sets really have quite useful properties.

**Proposition 121.** *Let $(E, d)$ be a metric space.*

1. *If $E$ is compact and $A \subseteq_C E$, then $A \subseteq_K E$.*
2. *If $A \subseteq_K E$, then $A \subseteq_C E$ and $A$ is bounded.*

**Proof.**

1.

2.

∎

Compactness is a **topological notion**, unlike completeness.

**Proposition 122.** *Let* $(E,d)$ *and* $(F,\delta)$ *be metric spaces, together with a continuous function* $f:(E,d)\to(F,\delta)$*. If* $A\subseteq_K E$ *then* $f(A)\subseteq_K F$*.*

**Proof.**

$\blacksquare$

**Proposition 123.** *Let $f : (E, d) \to (F, \delta)$ be a continuous bijection. If $(E, d)$ is compact, then $f$ is a homeomorphism.*

**Proof.**

$\blacksquare$

Perhaps the most famous theorem linking continuous functions and compact spaces is the result to which we were alluding to at the start of this section.

**Proposition 124.** (Min-Max Theorem)
*Let $f : (E, d) \to \mathbb{R}$ be continuous. If $(E, d)$ is compact, then $f$ is bounded and $\exists \mathbf{a}, \mathbf{b} \in E$ such that $f(\mathbf{a}) = \inf_{\mathbf{x}\in E} f(\mathbf{x})$ and $f(\mathbf{b}) = \sup_{\mathbf{x}\in E} f(\mathbf{x})$.*

**Proof.**

■

The next result cannot be generalized to infinite dimensional spaces (such as with $\ell^2(\mathbb{N})$ or other infinite dimensional Banach spaces).

**Proposition 125.** (Heine-Borel)
*Any closed bounded subset of* $\mathbb{K}^n$ *is compact in the usual topology.*

**Proof.**

∎

# 10.2 – Connected Spaces

Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a continuous function such that $\exists a, b \in A$ with $f(a)f(b) < 0$. What condition do we need on $A$ in order to guarantee the existence of a solution to $f(x) = 0$ on $A$?

Whether $A$ is compact or not is irrelevant: for instance, in the standard topology, the function $f : A = [0, 1] \cup [2, 3] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} -1 & x \in [0, 1] \\ 1 & x \in [2, 3] \end{cases}$$

is continuous over the compact set $A$, there are points $a, b \in A$ such that $f(a)f(b) < 0$, yet $f(x) \neq 0$ for all $x \in A$.

On the other hand, $f : A = [-1, 1] \to \mathbb{R}$ defined by $f(x) = x$ is such that $f(-1)f(1) < 0$ and $\exists x \in A$ such that $f(x) = 0$ (namely, $x = 0$).

The key notion is that of **connectedness**.

Let $(E, d)$ be a metric space. A **partition** of $E$ is a collection of two disjoint non-empty subsets $U, V \subseteq E$ such that $E = U \cup V$.

We denote the disjoint union by $E = U \sqcup V$.

An **open partition** of $E$ is a partition where $U, V \subseteq_O E$; a **closed partition** of $E$ is a partition where $U, V \subseteq_C E$.

# Examples:

1.

2.

3.

**Proposition 126.** *Let $(E, d)$ be a metric space. The following conditions are equivalent:*

1. *$E$ has no open partition;*

2. *$E$ has no closed partition;*

3. *The only subsets of $E$ that are both open and closed are $\varnothing$ and $E$ (such sets are rather unfortunately known as **clopen sets**).*

**Proof.**

$\blacksquare$

A metric space $(E, d)$ is said to be **connected** if it satisfies any of the conditions listed in Proposition 126.

Similarly, a subset $A \subseteq E$ is **connected** if its only clopen partition is trivial, that is: whenever $A = X \sqcup Y$, $X, Y \subseteq_O E$, either $X = \varnothing$ or $Y = \varnothing$.

We will denote such a situation with $A \subseteq_{©} E$, which is emphatically not a notation you will find anywhere else.

## Examples:

1. In the usual topology, $\mathbb{R}$ is .

2. In the usual topology on $\mathbb{R}$, $A = [0, 1] \cup [2, 3]$ is
   .

3. The singleton set $E = \{*\}$ is .

4. Show that $A = \{\frac{1}{n} \mid n \in \mathbb{N}\}$ is not a connected subspace of $\mathbb{R}$ in the usual topology.

   **Proof.**

   $\blacksquare$

As was the case with compactness, connectedness is a topological notion.

**Proposition 127.** *Let* $f : (E,d) \to (F,\delta)$ *be continuous. If* $A \subseteq_{©} E$*, then* $f(A) \subseteq_{©} F$*.*

**Proof.**

$\blacksquare$

# 10.2.1 – Characterization of Connected Spaces

We now give a simple necessary and sufficient condition for connectedness. Throughout, we endow the set $\{0, 1\}$ with the discrete metric.

**Proposition 128.** *A metric space $(E, d)$ is connected if and only if every continuous function $f : E \to \{0, 1\}$ is constant.*

**Proof.**

■

In practice, Proposition 128 is typically easier to use to show that a space is not connected.

**Proposition 129.** *Let* $(E, d)$ *be a metric space and* $A \subseteq_{©} E$. *If* $B \subseteq E$ *is such that* $A \subseteq B \subseteq \overline{A}$, *then* $B \subseteq_{©} E$.

**Proof.**

■

There is a series of other useful propositions about connected spaces.

**Proposition 130.** *If $(B_i)_{i\in I}$ is a family of connected subsets of a metric space $(E,d)$ such that $\bigcap_{i\in I} B_i \neq \varnothing$, then $B=\bigcup_{i\in I} B_i \subseteq_{©} E$.*

**Proof.**

■

**Proposition 131.** *If $(C_n)_{n\in\mathbb{N}}$ is a sequence of connected subsets of a metric space $(E,d)$ such that $C_{n-1}\cap C_n\neq\varnothing$, then $C=\bigcup_{n\in\mathbb{N}}C_n\subseteq_{©} E$.*

**Proof.** ■

**Proposition 132.** *Let $(E_1,d_1),\ldots,(E_n,d_n)$ be metric spaces. Then*

$$(E,d)=(E_1\times\cdots\times E_n,\sup\{d_i\mid 1\leq i\leq n\})$$

*is connected if and only if $(E_i,d_i)$ is connected for all $i$.*

**Proof.** ■

Let $(E, d)$ be a metric space once more. We define an equivalence relation on $E$ as follows:

$$\mathbf{x}R\mathbf{y} \iff \exists C \subseteq_{\copyright} E \text{ such that } \mathbf{x}, \mathbf{y} \in C. \tag{3}$$

The equivalence class

$$[\mathbf{x}] = \{\mathbf{y} \in E \mid \mathbf{y}R\mathbf{x}\} = \bigcup_{\substack{C \subseteq_{\copyright} E \\ \mathbf{x} \in C}} C$$

is a connected subset of $E$, which we call the **connected component** of $\mathbf{x}$.

It is not hard to show that $[\mathbf{x}] \subseteq_C E$ and that if a metric space only has a finite number of connected components, then each of those components is a clopen subset of $E$ (see exercises 9 and 10).

**Proposition 133.** *Consider $\mathbb{R}$ with the usual topology. Then, $A \subseteq_{©} \mathbb{R}$ if and only if $A$ is an interval.*

**Proof.**

We can now give a general proof of the remark that was made after Theorem 36.

**Corollary 134.** (BOLZANO'S THEOREM)
*Consider $\mathbb{R}$ with the usual topology and a continuous function $f : \mathbb{R} \to \mathbb{R}$. The image of any interval by $f$ is an interval.*

**Proof.**

$\blacksquare$

# 10.2.2 – Path-Connected Spaces

Let $(E,d)$ be a metric space. We say that $E$ is **path-connected** if for any two points $\mathbf{x},\mathbf{y}\in E$, there is a continuous function $\gamma:[0,1]\to E$ such that $\gamma(0)=\mathbf{x}$ and $\gamma(1)=\mathbf{y}$.

The **segment between $\mathbf{x}$ and $\mathbf{y}$** is $[\mathbf{x},\mathbf{y}]=\{t\mathbf{x}+(1-t)\mathbf{y}\mid t\in[0,1]\}$.

The continuous function associated to this segment is the function $f_{\mathbf{x},\mathbf{y}}:[0,1]\to E$ defined by $f_{\mathbf{x},\mathbf{y}}(t)=t\mathbf{x}+(1-t)\mathbf{y}$.

If $[\mathbf{x},\mathbf{y}]$ and $[\mathbf{z},\mathbf{w}]$ are two segments, define their **sum** to be

$$[\mathbf{x},\mathbf{y}]+[\mathbf{z},\mathbf{w}]=\{2t\mathbf{x}+(1-2t)\mathbf{y}\mid t\in[0,\tfrac{1}{2}]\}\cup\{(2t-1)\mathbf{z}+(2-2t)\mathbf{w}\mid t\in[\tfrac{1}{2},1]\}.$$

If $\mathbf{y} = \mathbf{z}$, the continuous function associated to this sum of segment is the function $g_{\mathbf{x},\mathbf{y},\mathbf{w}} : [0,1] \to E$ defined by

$$g_{\mathbf{x},\mathbf{y},\mathbf{w}}(t) = \begin{cases} 2t\mathbf{x} + (1-2t)\mathbf{y} & \text{if } t \in [0, \frac{1}{2}] \\ (2t-1)\mathbf{y} + (2-2t)\mathbf{w} & \text{if } t \in [\frac{1}{2}, 1] \end{cases}$$

**Examples:**

1. In $(\mathbb{R}^2, d_2)$, $B(\mathbf{0}, 1)$ is path-connected.

   **Proof.**

   $\blacksquare$

2.

**Proposition 135.** *If $(E, d)$ is path-connected, then it is also connected.*

**Proof.**

∎

**Proposition 136.** *If* $A \subseteq_{©} \mathbb{K}^n$ *in the usual topology, then* $A$ *is path-connected.*

**Proof.** ■

In general, connected spaces are not path-connected (see Problem 25), although there are many instances when they are.

**Theorem 137.** *Let* $(E, \|\cdot\|)$ *be a normed vector space over* $\mathbb{K}$*. Then any* $A \subseteq_{O,©} E$ *is path-connected.*

**Proof.**

**Proposition 138.** *Let $f : (E, d) \to (F, \delta)$ be a continuous map. If $E$ is path-connected, then $f(E)$ is path-connected.*

**Proof.** ■

# 10.3 – Exercises

1. Show that any compact metric space is precompact and complete.
2. Show that any complete precompact metric space is compact.
3. Prove Theorem 120.
4. With the usual metric, show that $A \subseteq \mathbb{R}^n$ is precompact if and only if $\overline{A} \subseteq_K \mathbb{R}^n$.
5. Prove Proposition 131.
6. Prove Proposition 132.
7. Let $(E_1, d_1), \ldots, (E_n, d_n)$ be metric spaces. Show that

$$(E, d) = (E_1 \times \cdots \times E_n, \sup\{d_i \mid 1 \leq i \leq n\})$$

is compact if and only if $(E_i, d_i)$ is compact for all $i = 1, \ldots, n$. [This result cannot be generalized to infinite products (**Tychonoff's Theorem**) without calling upon the **Axiom of Choice**, a.k.a **Zorn's Lemma**, a.k.a. the **Existence of Non-Measurable Sets**, a.k.a. the **Banach-Tarksi Paradox**.]

8. Show that (3) defines an equivalence relation on a metric space $(E, d)$.
9. Let $(E, d)$ be a metric space and let $\mathbf{x} \in E$. Show that $[\mathbf{x}] \subseteq_C E$.
10. Let $(E, d)$ be a metric space with finitely many connected components. Show that each of those components is a clopen subset of $E$.
11. Prove Proposition 136.
12. Show that if $(E, \|\cdot\|)$ is a normed vector space over $\mathbb{K}$, then any open ball $B(\mathbf{x}, \rho)$ is path-connected.
13. Let $(E, d)$ be a metric space, $B \subseteq_{\copyright} E$ and $A \subseteq E$ such that

$$B \cap \mathrm{int}(A) \neq \varnothing \quad \text{and} \quad B \cap \mathrm{int}(E \setminus A) \neq \varnothing.$$

Show that $B \cap \partial A \neq \varnothing$.
14. Let $(A, d_1)$ and $(B, d_2)$ be two metric spaces. Let $X \subsetneq A$ and $Y \subsetneq B$. Show that

$$(A \times B) \setminus (X \times Y) \subseteq_{\copyright} A \times B.$$

15. Prove Proposition 138.

16. In the usual topology, give an example of a subset $A \subseteq_{©} \mathbb{R}^2$ for which $\text{int}(A)$ is not connected.
17. In the usual topology, give an example of a subset $A \subseteq \mathbb{R}^2$ for which $\overline{A} \subseteq_{©} \mathbb{R}^2$ but $A$ is not connected.
18. Show that if the connected components of a compact set are open, then there are finitely many of them.
19. Let $(E, d)$ and $(F, \delta)$ be metric spaces, together with a continuous map $f : E \to F$ such that $f_{-1}(W) \subseteq_K E$ for all $W \subseteq_K F$. Show that $f$ is a closed map.
20. Let $(E, d)$ be a metric space.
    (a) If $W_1, W_2 \subseteq_K E$, show that $\exists \mathbf{x}_i \in W_i$ such that $d(\mathbf{x}_1, \mathbf{x}_2) = d(W_1, W_2)$.
    (b) If $W \subseteq_K E$ and $F \subseteq_C E$ are such that $W \subseteq F = \varnothing$, show that $d(W, F) \neq 0$. Is the conclusion still valid when $W \subseteq_C E$ is not necessarily compact?

21. Let $(E,d)=(\mathbb{R}^n,d_2)$.
    (a) If $F\subseteq_C E$ is unbounded and $f:F\to\mathbb{R}$ is a continuous map such that
    $$\lim_{\|\mathbf{x}\|\to\infty} f(\mathbf{x})=+\infty,\qquad \mathbf{x}\in F,$$
    show $\exists \mathbf{x}\in F$ such that $f(\mathbf{x})=\inf_{\mathbf{y}\in F} f(\mathbf{y})$.
    (b) If $W\subseteq_K E$ and $F\subseteq_C E$, show $\exists \mathbf{x}\in W, \mathbf{y}\in F$ such that $d(\mathbf{x},\mathbf{y})=d(W,F)$. Is the conclusion still valid when $E$ is an infinite-dimensional vector space over $\mathbb{R}$?
22. Let $(E,d)$ be a compact metric space with a map $f:E\to E$ such that $\forall \mathbf{x}\neq\mathbf{y}\in E$, $d(f(\mathbf{x}),f(\mathbf{y}))<d(\mathbf{x},\mathbf{y})$.
    (a) Show that $f$ admits a unique fixed point $\alpha\in E$.
    (b) Let $\mathbf{x}_0\in E$. For each $n\in\mathbb{N}$, set $\mathbf{x}_{n+1}=f(\mathbf{x}_n)$. Show that $\mathbf{x}_n\to\alpha$.
    (c) Are these results still valid if $E$ is complete but not compact?
23. Let $(E,d)$ and $(F,\delta)$ be two metric spaces, together with a injective map $f:E\to F$. Show that $f$ is continuous if and only if $f(W)\subseteq_K F$ for all $W\subseteq_K E$.
24. Let $(E,d)$ be a connected metric space and let $F\subseteq_C E$, with $\partial F\subseteq_{©} E$. Show that $F\subseteq_{©} E$. Is the result still true if $F$ is not necessarily closed?

25. Let $\Gamma = \left[\bigcup_{x\in\mathbb{Q}}(\{x\} \times (0,\infty))\right] \cup \left[\bigcup_{x\in\mathbb{R}\setminus\mathbb{Q}}(\{x\} \times (-\infty,0))\right] \subseteq \mathbb{R}^2$.
    (a) Show that $\Gamma \subseteq_{©} \mathbb{R}^2$.
    (b) Show that $\Gamma$ is not path-connected.
26. Let $(E,d)$ be a metric space. If $\varepsilon > 0$, we say that $E$ is $\varepsilon-$**chained** if for all $\mathbf{a},\mathbf{b} \in E$, $\exists n \in \mathbb{N}^\times$ and $\mathbf{x}_0,\ldots,\mathbf{x}_n \in E$ such that $\mathbf{x}_0 = \mathbf{a}$, $\mathbf{x}_n = \mathbf{b}$ and $d(\mathbf{x}_i,\mathbf{x}_{i-1}) < \varepsilon$ for all $i = 1,\ldots,n$. We say that $E$ is **well-chained** if it is $\varepsilon-$chained for all $\varepsilon > 0$.
    (a) If $E$ is connected, show that $E$ is well-chained.
    (b) If $E$ is compact and well-chained, show that $E$ is connected. Is the result still true if $E$ is not necessarily compact?
27. Let $(E,d)$ be a compact metric space and let $(\mathbf{x}_n)_{n\in\mathbb{N}} \subseteq E$ be such that $d(\mathbf{x}_n,\mathbf{x}_{n+1}) \to 0$. Show that the set of limit points of $(\mathbf{x}_n)_{n\in\mathbb{N}}$ is connected.
28. Let $f : \mathbb{R} \to \mathbb{R}^2$ be a bijection. Show that $f$ cannot be a homeomorphism.
29. Prove Darboux's Theorem: let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function, not necessarily of class $C^1$. Let $\varnothing \neq I = (a,b) \subseteq \mathbb{R}$. Show that $f'(I)$ is an interval in $\mathbb{R}$ using the set
$$\Gamma = \left\{ \frac{f(x)-f(y)}{x-y} \,\middle|\, (x,y) \in I^2, x < y \right\}.$$

30. Let $(E, d)$ be a metric space, with two disjoint sets $A, B \subseteq_C E$. Show that there exists a continuous function $f : E \to [0, 1]$ such that $A = f^{-1}(\{0\})$ and $B = f^{-1}(\{1\})$, as well as two disjoint sets $U, V \subseteq_O E$ such that $A \subseteq U$ and $B \subseteq V$.

# Solutions

20. **Proof.**

    (a)

(b)

21. **Proof.**

(a)

(b)

22. **Proof.**

(a)

(b)

(c)

∎

23. **Proof.**

26. **Proof.**

(a)

(b)

30. **Proof.**