Mathematical Analysis

Chapter 11
Normed Vector Spaces

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Winter 2022

Overview

The main aim of this chapter is to show that linear transformations between finite-dimensional normed vector spaces (n.v.s.) over $\mathbb{K}$ are continuous.

Outline

11.1 – Normed Vector Spaces

Normed vector spaces were introduced in chapter 9.

Let $p \geq 1$ and $A \in \mathbb{M}_{m,n}(\mathbb{K})$. Define

$$\|A\|_p = \sup_{\|\mathbf{x}\|_p \leq 1} \|A\mathbf{x}\|_p.$$

It is not too hard to show that

$$\|A\|_\infty = \max_{1\leq i\leq m} \left\{ \sum_{j=1}^{n} |a_{ij}| \right\}, \quad \|A\|_1 = \max_{1\leq j\leq n} \left\{ \sum_{i=1}^{m} |a_{ij}| \right\} \qquad (1)$$

$$\|A\|_2 = \text{ largest singular value of } A \qquad (2)$$

The operations of a normed vector space behave extremely well.

Proposition 139. *Let E be a normed vector space over $\mathbb{K}$. The maps $+ : E \times E \to E$ and $\cdot : \mathbb{K} \times E \to E$ are continuous.*

Proof. ■

Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be normed vector spaces over $\mathbb{K}$.

A map $T : E \to F$ is **linear** if

$$T(\mathbf{0}_E) = \mathbf{0}_F \quad \text{and} \quad T(a\mathbf{x} + b\mathbf{y}) = aT(\mathbf{x}) + bT(\mathbf{y}),\ \forall a, b \in \mathbb{K}, \mathbf{x}, \mathbf{y} \in E.$$

The set of all linear maps from E to F is denoted by $L(E, F)$. For instance, if $E = \mathbb{K}^n$ and $F = \mathbb{K}^m$, then $L(E, F) \simeq \mathbb{M}_{m,n}(\mathbb{K})$.

Theorem 140. *Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be two normed vector spaces over $\mathbb{K}$ and let $f \in L(E, F)$. The following conditions are equivalent:*

1. *f is continuous over E*
2. *f is continuous at $\mathbf{0} \in E$*
3. *f is bounded over $\overline{B(\mathbf{0}, 1)}$*
4. *f is bounded over $S(\mathbf{0}, 1)$*
5. *$\exists M > 0$ such that $\|f(\mathbf{x})\|_F \leq M\|\mathbf{x}\|_E$ for all $\mathbf{x} \in E$.*
6. *f is Lipschitz continuous*
7. *f is uniformly continuous*

Proof.

■

If $f \in L(E,F)$ is continuous (that is, if $f \in L_c(E,F)$), it then makes sense to define

$$\|f\| = \sup_{\|\mathbf{x}\|_E=1} \|f(\mathbf{x})\|_F = \sup_{\|\mathbf{x}\|_E\leq 1} \|f(\mathbf{x})\|_F.$$

With this definition, $(L_c(E,F), \|\cdot\|)$ is a normed vector space.

Furthermore, if $f \in L_c(E,F)$ and $g \in L_c(F,G)$ then $g \circ f \in L_c(E,G)$ and we have

$$\|(g\circ f)(\mathbf{x})\| = \|g(f(\mathbf{x}))\| \leq \|g\|\|f(\mathbf{x})\| \leq \|g\|\|f\|\|\mathbf{x}\| \leq M\|\mathbf{x}\|$$

for some $M > 0$ and for all $\mathbf{x} \in E$. In particular, $\|f \circ g\| \leq \|f\|\|g\|$.

The composition thus defines a kind of multiplication on $L_c(E,E)$; together with this multiplication, $L_c(E,E)$ is a **normed algebra**.

Theorem 141. *If F is a Banach space over $\mathbb{K}$, then so is $L_c(E,F)$.*

Proof.

■

We have seen that the metrics d_p are equivalent in $\mathbb{K}^n$, for $p \geq 1$. Can the same be said about the norms?

In fact, we can say even more: not only are the $p-$norms equivalent, but all norms on $\mathbb{K}^n$ are equivalent.

Proposition 142. *Let E be a finite dimensional vector space over $\mathbb{K}$. All norms on E are equivalent.*

Proof.

■

In general, this result is not valid if E is infinite-dimensional.

Corollary 143. *Let E be a finite-dimensional vector space over $\mathbb{K}$ and let $(F, \|\cdot\|_F)$ be any normed vector space over $\mathbb{K}$. If $f : E \to F$ is a linear mapping, then f is continuous.*

Proof.

■

Corollary 144. *Any finite-dimensional vector space over $\mathbb{K}$ is a Banach space.*

Proof.

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Corollary 145. *Any finite-dimensional subspace of a normed vector space over $\mathbb{K}$ is closed.*

Corollary 146. *The compact subsets of a finite-dimensional normed vector are the subsets that are both closed and bounded under the norm.*

11.2 – Exercises

1. Show that (1) and (2) define norms over $\mathbb{M}_n(\mathbb{K})$.
2. Let E be a n.v.s. over $\mathbb{R}$ and $A, B \subseteq E$. Denote $A+B = \{\mathbf{a}+\mathbf{b} \mid (\mathbf{a}, \mathbf{b}) \in A \times B\}$.
   (a) If $A \subseteq_O E$, show that $A + B \subseteq_O E$.
   (b) If $A \subseteq_K E$ and $B \subseteq_C E$, show that $A + B \subseteq_C E$. Is the result still true if A is only assumed to be closed in E?
3. Let E be a normed vector space over $\mathbb{R}$ and $\varphi : E \to \mathbb{R}$ be a linear functional on E.
   (a) Show directly that φ is continuous on E if and only if $\ker \varphi \subseteq_C E$.
   (b) i. Let F be a subspace of E. Show that the map $N : E/F \to \mathbb{R}$ defined by

$$N([\mathbf{x}]) = \inf_{\mathbf{y} \in [\mathbf{x}]} \{\|\mathbf{y}\|\}$$

   is a **semi-norm** on the quotient space E/F. What more can you say if $F \subseteq_C E$?
   ii. Show part (a) again, using part (b)i.

4. Prove Proposition 139.
5. Prove Corollary 145.
6. Prove Corollary 146.
7. Let E be a normed vector space with a countably infinite basis. Show that E cannot be complete.
8. Let E be an infinite-dimensional normed vector space over $\mathbb{R}$. Show that $D(\mathbf{0}, 1)$ is not compact in E by showing that it is not pre-compact in E (by what name is this result usually known?).
9. If $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$, define $\|\mathbf{x}\|_\infty = \sup\{|x_1|, \ldots, |x_n|\}$. Show that $\mathbf{x} \mapsto \|\mathbf{x}\|_\infty$ defines a norm on $\mathbb{R}^n$.
10. Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and define the inner product $(\mathbf{x} \mid \mathbf{y}) = x_1y_1 + \cdots + x_ny_n$. As seen in class, this inner product defines a norm $\|\mathbf{x}\| = \sqrt{(\mathbf{x} \mid \mathbf{x})}$. Show the **Parallelogram Identity**: $\|\mathbf{x} + \mathbf{y}\|^2 + \|\mathbf{x} - \mathbf{y}\|^2 = 2(\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2)$, $\forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.
11. Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Is it true that $\|\mathbf{x} + \mathbf{y}\|_\infty = \|\mathbf{x}\|_\infty + \|\mathbf{y}\|_\infty$ if and only if $\mathbf{x} = c\mathbf{y}$ or $\mathbf{y} = c\mathbf{x}$ for some $c \geq 0$?

Solutions

2. **Proof.**

(a)

(b)

3. **Proof.**

(a)

(b) i.

ii.

9. **Proof.**

10. **Proof.**

11. **Solution.**